

Spin stiffness of vector spin glasses

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Abstract

We consider properties of domain-wall excitations for $O(m)$ vector spin glasses in the limit $m \rightarrow \infty$, where the energy landscape is simplified considerably compared to XY or Heisenberg models due to the complete disappearance of metastability. Using numerical ground-state calculations and appropriate pairs of complementary boundary conditions, domain-wall defects are inserted into the systems and their excitation energies are measured. This allows us to numerically determine the stiffness exponents for lattices of a range of spatial dimensions $d = 2, \dots, 6$. Compilation of these results finally enables an estimate regarding the lower critical dimension of the model. The results are compared to estimates resulting from field-theoretic calculations.

Keywords: vector spin glasses, spherical limit, lower critical dimension

1. Introduction

In view of the notorious difficulties in understanding the behavior of spin glass models of low spin dimension such as the Ising ($m = 1$), XY ($m = 2$) or Heisenberg ($m = 3$) models, it appears appealing to investigate the limiting case of vector spins with an infinite number of spin components ($m \rightarrow \infty$), which turns out to be well amenable to a numerical analysis. The model is described by the well-known Edwards-Anderson Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (1)$$

where the $\mathbf{S}_i \in \mathbb{R}^m, i = 1, \dots, N$, are vector spins with m components, here taken to be normalized as $|\mathbf{S}_i| = 1$. The exchange couplings J_{ij} are drawn independent of each other from a Gaussian probability distribution with zero mean and unit standard deviation. The relative numerical ease of handling this limiting case is due to a significant simplification of the energy landscape with increasing spin dimension, namely a reduction in the number of metastable states. This effect also leads to an increase in the numerically accessible lattice sizes in MC simulations, e.g., of Heisenberg spin glasses [1] as compared to the Ising case [2].

Ising [2, 3], XY [4] and Heisenberg [1, 5, 6] models have received most attention and it seems relatively clear now, that for all of them the lower critical dimension is $2 \leq d_l \leq 3$ [7], i.e., finite-temperature spin glass transitions are found in $d = 3$, but only zero-temperature transitions in $d = 2$. The situation is much less clear for general

numbers m of spin components and, in particular, it is an open issue whether the lower critical dimension depends on the number of spin components for general m and in the limit of $m \rightarrow \infty$.

Considering a finite system of N spins in the limit of large spin dimensions, metastability vanishes completely and the ground state becomes unique, occupying only a finite-dimensional sub-manifold in spin space [8, 9]. As a consequence, for each system size there exists a finite, critical number of spin components above which the ground-state energy does not change upon further adding spin dimensions, such that the system effectively describes a spherical spin glass, i.e. the limit $m \rightarrow \infty$. Compared to the field theoretic calculations [10, 11], this corresponds to an interchange of the limits $N \rightarrow \infty$ and $m \rightarrow \infty$, such that the thermodynamic limit $N \rightarrow \infty$ is taken first in the perturbative calculations, whereas the spherical limit $m \rightarrow \infty$ is taken first in the numerical approach. This leads to some subtleties, such that the numerical approach might be considered the zeroth-order term in a $1/m$ expansion around the field-theoretic calculation [12].

There is a strict upper bound on the number of spin components, beyond which no further change in the ground state is observed [9],

$$m_{\max}(N) = \left\lfloor \left(\sqrt{8N+1} - 1 \right) / 2 \right\rfloor \sim N^\mu, \quad \mu = 1/2, \quad (2)$$

where $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to x . For the fully connected or Sherrington-Kirkpatrick limit of the model, the average maximum number of occupied spin dimensions in the ground state scales with an exponent $\mu = 2/5$ [8]. For the short-range, nearest-neighbor models on hyper-cubic lattices considered here, on the other hand, the spins occupy only a somewhat smaller number of spin dimensions, with $\mu = \mu(d) < 2/5$

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depending on the lattice dimension d [13]. This situation allows for the limit of an infinite number of spin components to be studied numerically for finite systems using only a finite number of spin components, for instance determining the lower critical dimension from defect-energy calculations. Previously reported results [13, 14], however, did not reach up to the lattice dimensions above the apparent lower critical dimension, where an ordering effect would be expected at finite temperatures.

2. Ground state computations

We determine stiffness exponents for lattices of spatial dimensions $d = 2, \dots, 6$ numerically, using the concept of the defect energy [15]. This widely-used approach in the study of spin-glass systems is based on the assumption that the cost ΔE of the insertion of a system-size defect into a state of the ordered phase scales as [16]

$$\Delta E \propto L^\theta, \quad (3)$$

where θ is known as the spin-stiffness exponent. The significance of θ is based on the observation that for $\theta < 0$ arbitrarily small thermal fluctuations suffice to destroy the ordered state in the thermodynamic limit, preventing a phase transition to occur at $T > 0$. On the contrary, $\theta > 0$ indicates the presence of a finite-temperature phase transition, since the cost of insertion of a system-size defect diverges in the thermodynamic limit, thus guaranteeing a stable ordered phase at sufficiently small (but non-zero) temperatures. If, to leading order, the defect energy remains constant, i.e., $\theta = 0$, the system is at its lower critical dimension d_l .

The calculation of defect energies with a numerical approach requires the determination of ground states for a set of different boundary conditions. Ground states are determined here employing a local spin quench procedure, for which the spins are iteratively aligned with their respective local molecular fields $\mathbf{H}_i$, as

$$\mathbf{S}'_i \parallel \mathbf{H}_i = \sum_{j \in \mathcal{N}(i)} J_{ij} \mathbf{S}_j, \quad (4)$$

where the sum runs over the set $\mathcal{N}(i)$ of nearest neighbors of the spin at site i . It is easily seen that alignment of each spin with its molecular field is a necessary condition for the system to be in its ground state. For the present case of a system without metastable states [8], it is also sufficient.

These updates are interspersed with sweeps of over-relaxation moves to speed up convergence, which have also been found to improve the decorrelation of systems with finite m in Monte Carlo simulations [5]. These moves, again being local, preserve the energy of the whole spin configuration since the updated spin is merely rotated around its local field and therefore moves at constant energy. The

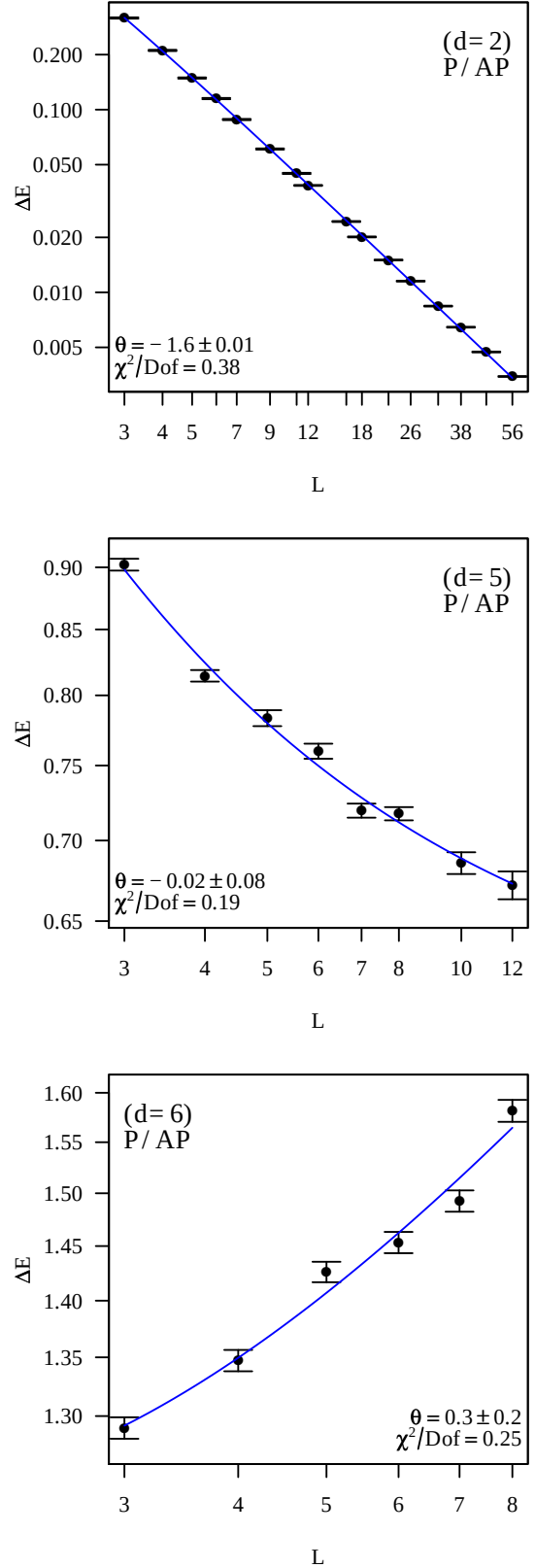


Figure 1: Defect energy scaling with lattice size L for the application of P/AP boundary conditions for $d = 2$ (top), $d = 5$ (middle) and $d = 6$ (bottom). At around $d = 5$ the spin-stiffness θ changes its character. Corrections to scaling around this dimension are most pronounced. The lines are fits of the functional form (6) to the data.

simplest way of implementing such a procedure, in particular for the case of arbitrary spin dimensions m , is to reflect the spin along $\mathbf{H}_i$, such that

$$\mathbf{S}'_i = -\mathbf{S}_i + 2 \frac{\mathbf{S}_i \cdot \mathbf{H}_i}{|\mathbf{H}_i|^2} \mathbf{H}_i. \quad (5)$$

This maximal movement can also be argued to lead to a maximal decorrelation effect within the constant-energy manifold of single-spin movements. The whole procedure of spin-quench and over-relaxation moves can be implemented very efficiently, since only a few elementary operations are required for each step, and no random numbers are involved.

Inducing a defect of size L in the system can be accomplished by manipulating the boundaries in *one* lattice dimension. To reduce additional surface effects, boundary conditions in all other lattice directions were chosen to be periodic. Here, we decided to use three different setups aimed at reducing problems with domain walls trapped due to the periodicity and possibly discriminating between spin (rotational symmetry) and chiral [17] (discrete symmetry) defects [18]. First, we investigated the case of periodic and anti-periodic (P/AP) boundary conditions as the standard setup to probe continuous or “spin” excitations. Since for this case, domain walls might be trapped in *both* configurations due to the imposed periodicity, the difference between the two ground-state energies does not directly correspond to the excitation energy of a single defect [19, 20]. To alleviate this problem, we also investigated open/domain-wall (O/DW) boundary conditions. In this case, the defect energy is determined by comparing the energies of a system with open boundaries in one direction and one having fixed boundary planes, in which the spins in one of the two hyperplanes are rotated around a common axis perpendicular to the hyperplane in spin space occupied by the boundary spins. Due to the open boundaries in the first case, the insertion of exactly one domain wall is guaranteed. A third type of boundary conditions (open/spin-pair) gave similar results compared to O/DW. These will be discussed elsewhere [21].

Due to the high precision of our results, we are able to also resolve corrections to the leading scaling behavior of Eq. (3). These become important, in particular, in the vicinity of the lower critical dimension, where θ vanishes. We find the corrections to be well described by a first-order analytic correction term resulting in the form

$$\Delta E = L^\theta (a + b/L). \quad (6)$$

We also considered the behavior of effective stiffness exponents as the aspect ratio of the systems is changed from unity to R , i.e., for considering systems of geometry $L^{d-1} \times RL$, $R = 1, 2, \dots$, which was observed to significantly reduce scaling corrections for Ising and XY spin glasses [20, 22]. Due to an asymptotically *exponential* decay of defect energies with system size in the one-dimensional spherical limit [14], however, this approach appears to be of only limited applicability here [21].

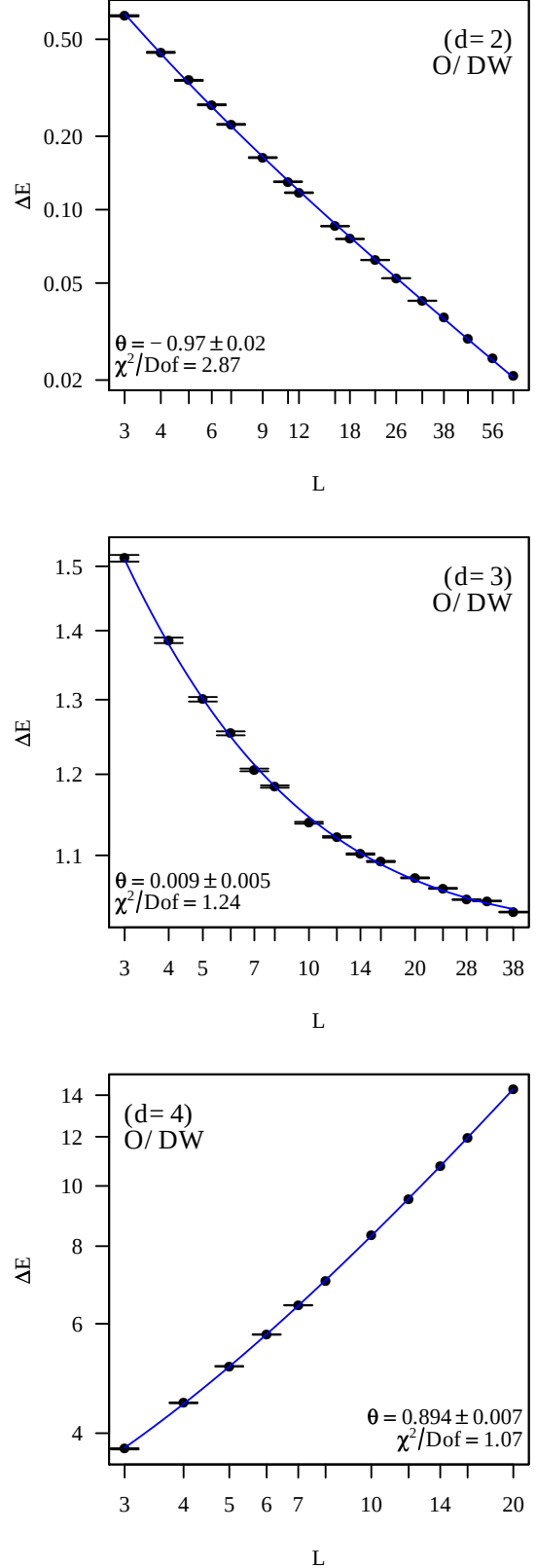


Figure 2: Defect energy scaling with lattice size L for the application of O/DW boundary conditions for $d = 2$ (top), $d = 3$ (middle) and $d = 4$ (bottom) with fits of the functional form (6) to the data. For this setup, the spin-stiffness exponent θ changes sign at around $d = 3$.

3. Results

Ground states have been computed here for a range of system sizes and the described sets of boundary conditions using 3000 disorder realizations per lattice size for hypercubic lattices of dimensions $d = 2, \dots, 6$. In Fig. 1 we present three showcase results for P/AP boundary conditions. We observe a change of the character of the spin stiffness at around $d = 5$. The mentioned finite-size corrections can be seen here quite well, since the deviation from a straight line, suggested by Eq. (3), appears to be rather strong in this logarithmic plot. Using a smaller range of lattice sizes, Lee and Young [13] were not able to resolve the corrections to scaling. This applies, in particular, to the marginal case $d = 5$, where only the extended range of system sizes reveals that the observed decay of defect energies settles down as the system size is increased and thus is due to corrections to scaling.

Figure 2 shows the results using O/DW boundary conditions. Here, remarkably, the spin stiffness changes its character already in $d = 3$, again going along with pronounced corrections to scaling. One might wonder whether the apparently rather different estimates of stiffness exponents from the two boundary setups could be due to them exciting different types of defects, for instance of the spin and chiral type. While the O/DW construction certainly forms the cleaner setup in that one directly probes the energy of a single domain-wall defect, distinctions between spin and chiral excitations are subtle for this model: in general, we would associate the sign of the determinant of the considered $O(m)$ rotation matrix with the chiral degrees of freedom [23]. Since each system is simulated with a number of spin components m somewhat larger than the number $m_{\max}$ actually occupied by the spins in their ground state, reflection operations on the unoccupied directions in spin space formally change the sign of the rotational determinant, but leave the actual effect on the spins invariant. Hence, the association of the O/DW setup to the excitation of chiral defects is not completely justified at this point. Note also that the authors of Ref. [24] suggest from ground-state calculations of three-dimensional Heisenberg spin glasses that stiffness exponents from the P/AP setup do not correspond to the intrinsic stiffness of spins.

In view of our results for the two different setups of boundary conditions, summarized in Fig. 3, it would be desirable to also study this model at finite temperatures, either by means of Monte Carlo simulations, or iterating equations derived from a saddle-point calculation as in Refs. [12, 25]. The results of such calculations will be reported elsewhere [21]. As for now, the lower critical dimension for spin ordering for the spherical spin glass appears to be 5, whereas the lower critical dimension for the chiral sector might be 3, see Fig. 3. These results do not support the conjecture $d_l = 8$ of [11], based on a perturbation expansion. One should keep in mind, however, the interchange of the limits $N \rightarrow \infty$ and $m \rightarrow \infty$, implicitly

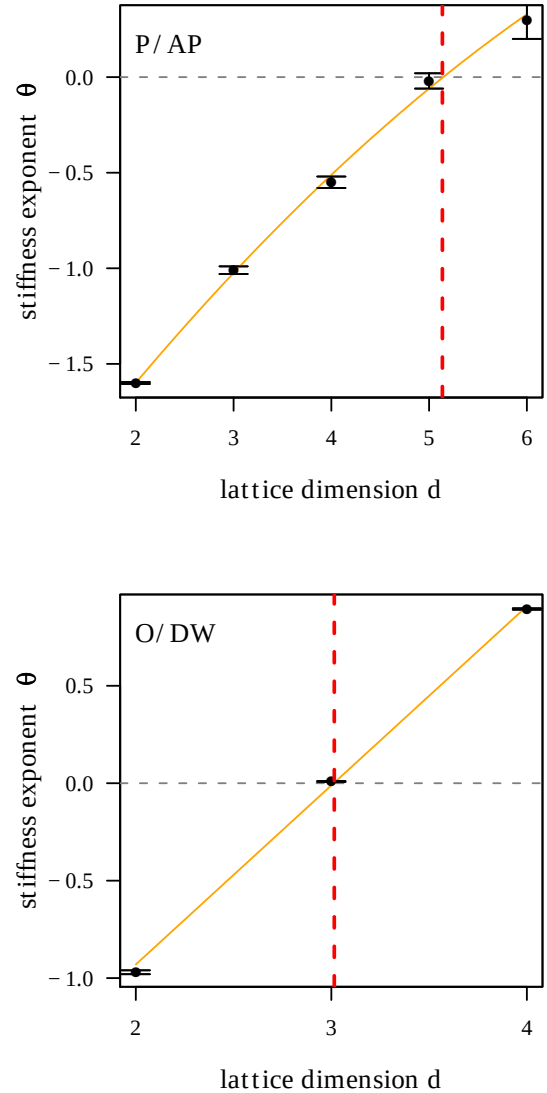


Figure 3: The evolution of the spin stiffness exponent θ with lattice-dimension d for P/AP boundary conditions (top) and for O/DW boundary conditions (bottom). The lower critical dimension can be found where the curve crosses $\theta = 0$. All lines are guides to the eye.

performed in numerical simulations. The upper critical dimension, on the other hand, is predicted to be 8 [10].

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