

Domain walls and Schramm-Loewner evolution in the random-field Ising model

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The concept of Schramm-Loewner evolution provides a unified description of domain boundaries of many lattice spin systems in two dimensions, possibly even including systems with quenched disorder. Here, we study domain walls in the random-field Ising model. Although, in two dimensions, this system does not show an ordering transition to a ferromagnetic state, in the presence of a uniform external field spin domains percolate beyond a critical field strength. Using exact ground state calculations of very large systems, we examine ground state domain walls near this percolation transition finding strong evidence that they are conformally invariant and satisfy the domain Markov property, implying compatibility with Schramm-Loewner evolution (SLE_κ) with parameter $\kappa = 6$. These results might pave the way for new field-theoretic treatments of systems with quenched disorder.

In the past decades, analytic techniques such as conformal field theory (CFT) and Coulomb gas methods have led to a rather comprehensive understanding of critical phenomena in two dimensions (2D). In particular, CFT allows for a complete classification of 2D critical points, the exact determination of critical exponents and, in some cases, even scaling amplitudes [1]. This success is tied to the fact that the conformal group is infinite-dimensional, however, which is true only in 2D, and few of the results generalize to higher dimensions [2]. Another difficulty for this approach arises for the important class of systems with quenched disorder, such as diluted magnets, random-field systems and spin glasses [3], since the non-unitary CFTs with central charge $c = 0$ that are believed to describe systems with quenched disorder are poorly understood [4].

Essential geometrical aspects of critical phenomena, which are not well understood in the framework of CFT, have recently been found to be described by a stochastic description of domain boundaries in terms of random curves in the plane in a framework dubbed Schramm-Loewner evolution (SLE) [5]. In SLE, stochastic curves in the plane are constructed from one-dimensional Brownian motion, thus classifying a statistical ensemble of curves with only one parameter, the diffusion constant κ . Characteristic interfaces in many physical systems have been shown (in some cases rigorously) to satisfy SLE_κ. These include percolation ($\kappa = 6$), self avoiding walks ($\kappa = 4/3$), as well as spin cluster boundaries ($\kappa = 3$) and Fortuin-Kasteleyn cluster boundaries ($\kappa = 16/3$) in the Ising model. In recent years, close connections between SLE and CFT, including links between probabilistic properties of SLE curves and scaling operators in CFT, or between the central charge c of the CFT and the diffusion constant κ have been established [5]. A number of numerical studies have found interfaces in disordered systems to be (partially) consistent with SLE, in particular the 2D Ising spin glass [6, 7], the Potts model on dynamical triangulations [8], the random bond Potts model [9], and the disordered solid-on-solid model [10]. Such findings and the close link between SLE and CFT spur the hope of a more complete understanding of systems with quenched disorder from a field-theoretic perspective.

Here, we study domain walls in the random-field Ising

model (RFIM) with Hamiltonian [11]

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - \sum_i h_i s_i, \quad (1)$$

where the spins $s_i = \pm 1$ are located on the sites of a square lattice and interact ferromagnetically with nearest neighbors. The local fields h_i are quenched random variables which, for the time being, we take as drawn from a Gaussian distribution with mean H and standard deviation Δ . Since only the ratio J/Δ is relevant, we take $J = 1$. Random field models have a large number of experimental realizations which are of technological importance such as superfluid helium, liquid crystals in silica aerogels, Bragg glasses in high- T_c superconductors, amorphous solids [12] and ferroelectric materials [13]. It was shown by Imry and Ma [14] that random fields destabilize the ferromagnetic order in dimensions $d \leq 2$. For the case of 2D, it was argued by Binder [15] that ferromagnetic order occurs only up to a break-up length scale $L_b \sim e^{A/\Delta^2}$, which grows with decreasing disorder Δ , and that the system remains paramagnetic at scales $L > L_b$. Later, Aizenman and Wehr [16] proved that for $d \leq 2$ the system indeed has a unique Gibbs state, precluding the existence of an ordering transition. On the contrary, for $d \geq 3$ L_b diverges at the thermodynamic transition point, below which the system is ferromagnetic [11]. For non-zero average fields H , on the other hand, the size of spin clusters diverges at a critical value $H_c = H_c(\Delta)$ [17–20]. However, the weight of these clusters is sub-extensive, such that the free energy remains analytic and no thermodynamic phase transition occurs. This phenomenon bears many similarities to the Kertész line in ferromagnets in the absence of disorder [21].

To study the properties of domain walls in the RFIM we numerically compute exact ground states of samples of random-field realizations. Ground states can be found in polynomial time via a mapping to a minimum-cut/maximum-flow problem [22]. We employ a fast algorithm based on the idea of “augmenting paths” [23], which allows us to find ground states of systems of 10^7 spins in about 6 s, such that the maximum system sizes exceed those of previous studies [17–19] by about an order of magnitude. We use a variety of domain geometries, partially with fixed spins to enforce the occurrence of domain walls.

In the framework of Loewner evolution one imagines a random curve γ_t in the plane as being continuously grown in time t in a random process. Instead of studying this process directly, one considers the evolution of a family $g_t : \mathbb{H} \setminus \gamma_t \rightarrow \mathbb{H}$ of conformal maps that take the complement of γ_t in the upper half plane $\mathbb{H}$ to $\mathbb{H}$. Under this map γ_t , which lies on the boundary of $\mathbb{H} \setminus \gamma_t$, is taken to the boundary of $\mathbb{H}$, i.e. to the real line. Assuming standard normalization and boundary conditions, it turns out that g_t is completely determined by the one-dimensional function ξ_t , which corresponds to the image under g_t of the tip of the growing curve on the real line, via the differential equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}. \quad (2)$$

It was shown by Schramm [24] that if the ensemble of curves γ_t is conformally invariant and satisfies the domain Markov property (to be discussed below), the one-dimensional random process described by ξ_t must be Brownian motion with zero mean and variance κt , where κ is the diffusion constant. For such SLE curves, many stochastic properties can be calculated rigorously including, for instance, the fractal dimension $d_f = 1 + \kappa/8$ or the probability $P_{LP}(x, y)$ that the curve γ_t passes to the left of the point (x, y) . The latter was proven by Schramm [25] for curves starting at the origin of the upper half plane $\mathbb{H}$ to be

$$P_{LP}^\kappa(x, y) = \frac{1}{2} + \frac{\Gamma(4/\kappa)}{\sqrt{\pi} \Gamma(\frac{8-\kappa}{2\kappa})} \frac{x}{y} {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}; \frac{3}{2}; -\left(\frac{x}{y}\right)^2\right) \quad (3)$$

where ${}_2F_1$ is Gauss' hypergeometric function.

The domain Markov property (DMP) formalizes the notion that a growing path of the type described above is agnostic about its past. Let $P_{\mathbb{D}}(\gamma_{ab})$ be the probability measure of curves γ_{ab} in a domain $\mathbb{D}$ running between points a and b on the boundary of $\mathbb{D}$, and let c be a point in the interior of $\mathbb{D}$. Then, the DMP states that

$$P_{\mathbb{D}}(\gamma_{cb}|\gamma_{ac}) = P_{\mathbb{D} \setminus \gamma_{ac}}(\gamma_{cb}), \quad (4)$$

i.e., the probability of γ_{cb} is independent of whether γ_{ac} is preconditioned in domain $\mathbb{D}$, or whether γ_{ac} is excluded from the domain itself. While it is not debated that the DMP holds for the scaling limit of domain boundaries in pure lattice systems [5], even off criticality, it has been argued that it is likely not to survive the average over quenched disorder and it is unclear, in general, whether it would be restored in the scaling limit [7]. This is due to the fact that the conditional probability on the lhs of (4) is a ratio of probabilities, and hence becomes in the presence of quenched disorder

$$P_{\mathbb{D}}(\gamma_{cb}|\gamma_{ac}) = \frac{P_{\mathbb{D}}(\gamma_{cb} \cap \gamma_{ac})}{P_{\mathbb{D}}(\gamma_{ac})} = \frac{[P_{\mathbb{D}}^i(\gamma_{cb} \cap \gamma_{ac})]_{h_i}}{[P_{\mathbb{D}}^i(\gamma_{ac})]_{h_i}}, \quad (5)$$

where $[\cdot]_{h_i}$ denotes an average over the randomness. The situation is simplified, however, at zero temperature where, with the unique ground state of the Gaussian distribution of fields, $P_{\mathbb{D}}^i(\gamma_{cb} \cap \gamma_{ac})$ and $P_{\mathbb{D}}^i(\gamma_{ac})$ are either 0 or 1 and hence

$P_{\mathbb{D}}(\gamma_{cb}|\gamma_{ac})$ is simply the ratio of the fraction of disorder samples that realize the corresponding curve segments. If the interface is induced by an appropriate choice of spin orientations along the boundary of the domain $\mathbb{D}$, the DMP survives the disorder average unperturbed: for calculating $P_{\mathbb{D}}(\gamma_{cb}|\gamma_{ac})$ the spins along the boundary of $\mathbb{D}$ are *asserted* to be either up or down in order to induce the interface. Meanwhile, the spins along the curve γ_{ac} are *conditioned* to be either up or down. For calculating $P_{\mathbb{D} \setminus \gamma_{ac}}(\gamma_{cb})$, on the other hand, the same spins are fixed in the same way, because the boundary of the domain runs along the curve γ_{ac} . For ground states (but not for finite temperatures), calculating disorder averages with fixed spins is equivalent to disorder averages with fixed and large fields. Therefore, configurations with the spins in question fixed will occur in the same fraction of disorder samples for both sides of (4), and the DMP holds exactly.

We examined the agreement of the RFIM interfaces with the SLE expectations for the left passage probabilities. As the rigorous result of Eq. 3 is valid on the upper half plane $\mathbb{H}$, we performed our ground state calculations on lattices embedded in domains $\mathbb{D}$ which have simple, closed-form conformal maps $w(z)$ to $\mathbb{H}$ [10]. By fixing the boundary spins through the respective random fields, the interface was forced to run between points a and b in $\mathbb{D}$ which are mapped to the origin and infinity in $\mathbb{H}$, respectively. Numerical checks were performed for the unit circle with the interface between $-i$ and i , which is mapped to $\mathbb{H}$ by $w(z) = i \frac{1+z}{1-z}$, as well as the unit square with interfaces defined from 0 to $1+i$, which is mapped to $\mathbb{H}$ by $w(z) = -\wp(1+i-z; 1, i)$ where $\wp(z; w_1, w_2)$ is the Weierstrass p-function. Looking at the left passage probability for multiple domains additionally acts as a check of conformal invariance. For a quantitative comparison we considered the mean square deviation of the computed left passage probability P_{LP} from the exact result P_{LP}^κ of Eq. (3),

$$E(\kappa) = \left\langle [P_{LP}(x, y) - P_{LP}^\kappa(x, y)]^2 \right\rangle_{\mathbb{D}}^{1/2}, \quad (6)$$

where $\langle \cdot \rangle_{\mathbb{D}}$ denotes a spatial average over $\mathbb{D}$, excluding the vicinity of the fixed boundary spins. This quantity is shown in Fig. 1 as a function of κ for the circle and square geometries. In both cases $E(\kappa)$ is minimal for κ within 0.05 of the value $\kappa = 6$. The spatial dependence of the deviation from $P_{LP}^{\kappa=6}(x, y)$ is also shown in Fig. 1. There appears to be no systematic deviation.

From the presented results it is evident that spin domain interfaces in the 2D RFIM satisfy the domain Markov property and are conformally invariant at the percolation threshold $H = H_c(\Delta)$, and thus are described by SLE. Furthermore, the parameter κ appears to be consistent with $\kappa = 6$ to high precision. To corroborate these findings, we tested our results for compatibility with the exact formulas for the crossing probabilities of percolation clusters, another system with $\kappa = 6$ [26]. These give the probability of finding a cluster of a given species (say up spins) which touches two non-adjacent segments of the boundary of a domain. In particular, the probability π_r of a domain touching both the top and bottom of a rectangle of aspect ratio r is known to depend only on r [26]. Similarly, in a domain defined by an equilateral triangle,

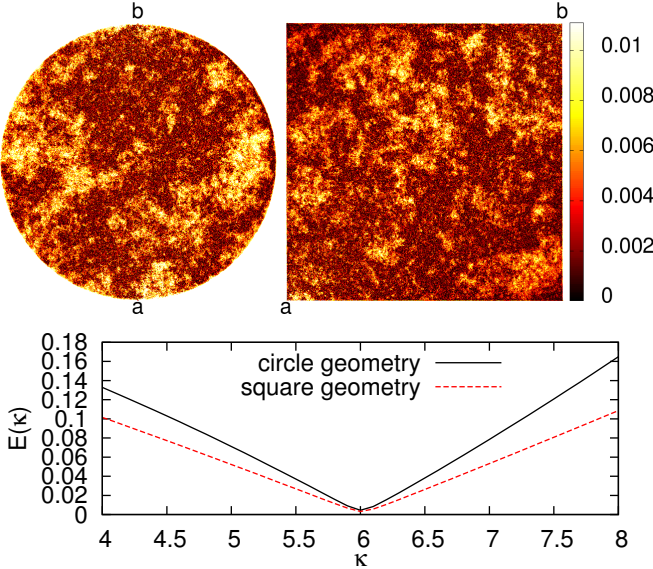


FIG. 1. Magnitude of deviation of the left passage probability for spin domain interfaces from the exact result of Eq. (3) for $\kappa = 6$ in circular and square domains. The interfaces are constrained to run between points a and b as shown. The lower panel displays the spatially averaged deviation (i.e. Eq. (6)) as a function of the diffusion constant κ , showing a clear minimum close to $\kappa = 6$. Calculations were performed for 10 000 disorder realizations on systems of 6×10^6 spins at $\Delta = 2.5$ and $H = H_c(\Delta) = 1.36 \times 10^{-2}$.

the probability of a cluster crossing from a fraction x of one boundary edge to the opposite edge is $\pi_x = x$ [27] (see Fig. 2 for a schematic representation). Figure 2 shows these exact percolation results together with numerical simulation data for the RFIM at $\Delta = 1.7$ and $H = 4.71 \times 10^{-4} \approx H_c(\Delta = 1.7)$ for a system of 6×10^6 spins for both the rectangle and triangle geometries, calculated from the ground states of 30 000 disorder samples per data point. We find very good agreement with the percolation results with only rather small systematic deviations indicating a possible small deviation from the true transition field $H_c(\Delta)$.

One of the rigorous results for curves described by SLE_κ is their fractal dimension, which is given, for $\kappa \leq 8$, by $d_f = 1 + \kappa/8$. We numerically determined the fractal dimension for a range of different geometries. We find that corrections to scaling for the interface length L_I are well described by the form

$$L_I = aL^{d_f}(1 + b/L). \quad (7)$$

We take L as the square root of the number of spins for the case of non-rectangular domains. For the circle geometry we find $d_f = 1.7506(9)$, while for the square geometry we arrive at $d_f = 1.7514(14)$. Both results are perfectly compatible with $d_f = 7/4 = 1.75$ expected for SLE curves with $\kappa = 6$.

As the most direct test for SLE , we studied the one-dimensional stochastic process (or driving function) ξ_t generated by the Loewner map g_t according to Eq. (2) as applied to domain walls in the RFIM. For a lattice system, the family of maps g_t is realized as a discrete series of maps g_i iteratively

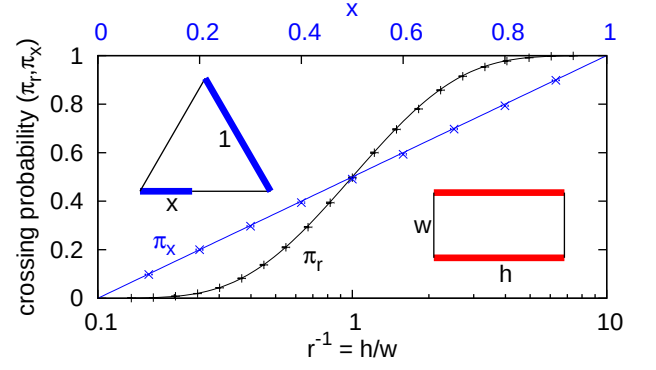


FIG. 2. Crossing probabilities for spin clusters in the 2D RFIM at $\Delta = 1.7$ and $H = 4.71 \times 10^{-4} \approx H_c(\Delta)$ in rectangular (lower scale) and triangular (upper scale) domains. The solid lines indicate the exact results for percolation clusters ($\kappa = 6$) derived in Refs. [26, 27]. Error bars are smaller than the symbols.

removing a small section from the beginning of the curve. For this purpose, g_i is approximated using a vertical slit map [28]

$$g_i(z) = i\sqrt{-(z - \xi_i)^2 - 4\Delta t_i} + \xi_i. \quad (8)$$

Here, ξ_i and Δt_i are determined through $\xi_i = x_{i,i-1}$ and $\Delta t_i = y_{i,i-1}^2/4$, where $x_{i,i-1}$ and $y_{i,i-1}$ are the coordinates of the i 'th segment of the curve after undergoing the $i-1$ successive maps $g_{i-1} \circ \dots \circ g_1$. The parameter ξ_i is the value of the driving function ξ_t sampled at time $t_i = \sum_{j \leq i} \Delta t_j$. The complex square root in Eq. (8) is calculated, as usual, with the branch cut along the negative real axis. We studied the statistics of 10 000 interfaces generated in a half disc, optimally mimicking the full space $\mathbb{H}$. The interface is initiated at the origin by two fixed spins and is considered ended when it touches the curved boundary. We used systems of 6 million spins at $\Delta = 2.5$ and $H = 1.36 \times 10^{-2} \approx H(\Delta)$. We find that the variance of the driving function calculated from the interfaces is $\hat{\kappa} = \langle (\xi_t - \langle \xi_t \rangle)^2 \rangle / t = 6.086(87)$, and the normalized mean is $\hat{\xi} = \langle \xi_t \rangle / \sqrt{\hat{\kappa}t} = 0.017(10)$, perfectly compatible with $\text{SLE}_{\kappa=6}$. Using a Kolmogorov-Smirnov test, we further checked that the ξ_t are normally distributed and find a p -value of $p = 0.17$, indicating consistency with a normal distribution. To check for the statistical independence of the increments of ξ_t , we divided ξ_t into n increments evenly spaced in time and checked whether the signs of these increments follow a χ^2 distribution with $2^n - 1$ degrees of freedom [28]. For $n = 10$, we find a p -value of $p = 0.18$ indicating consistency with the assumption of statistical independence. As a further check of conformal invariance, we performed the same tests on interfaces originating in the two domains of Fig. 1, and found similar agreement with SLE .

Finally, we also considered a binary (Bernoulli) field distribution, where each local random field h_i takes on the value Δ with probability p and $-\Delta$ with probability $1 - p$. Because of the discrete nature of the distribution, this system has a massive ground-state degeneracy, and behaves rather differently from the Gaussian RFIM, at least at zero temperature. Although polynomial algorithms for enumerating all ground

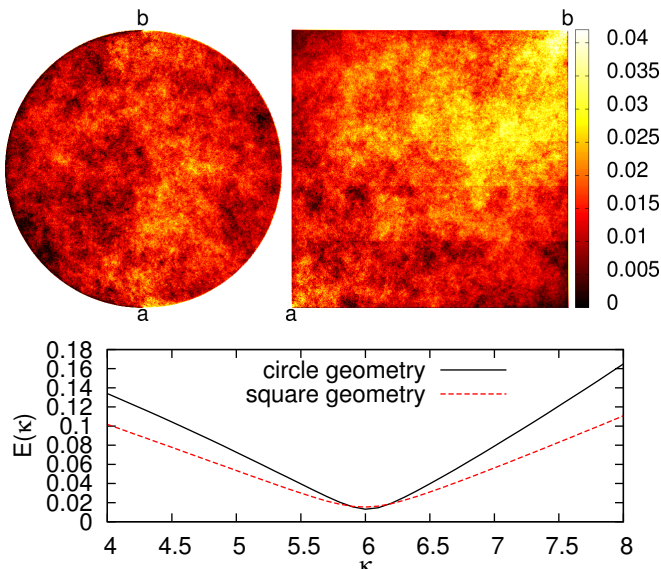


FIG. 3. Left passage probabilities for the binary distribution of fields. See Fig. 1 for a detailed description of the setup. The calculations were carried out at $\Delta = 2.1$ and $p = 0.522 \approx p_c(\Delta)$ for systems of 6 million spins. Averages were performed over 10 randomly chosen ground states for each of 6000 disorder configurations.

states are known [29], handling all ground states becomes impractical for larger system sizes. Here, instead, we sample from the ground-state manifold by adding a tiny noise term (normally distributed with strength δ) to the Bernoulli field distribution. For sufficiently small δ , the resulting state is also a ground state of the noise-less system, and, importantly, is selected without bias from among the degenerate ground states. We find that there exists a geometric transition where spin clusters diverge at $p = p_c(\Delta)$, in analogy with $H_c(\Delta)$ for the Gaussian case. Interestingly, for $\Delta \leq 2$ this transition ap-

pears to occur at the constant value $p_c(\Delta \leq 2) = 1/2$, while $p_c(\Delta > 2) > 1/2$. We test agreement with SLE predictions by looking at fractal dimension, finding $d_f = 1.746(2)$, and by looking at the left passage probability, the results of which are shown in Fig. 3. Both properties are consistent with SLE for $\kappa = 6$.

We have studied the properties of spin cluster interfaces in the ground state of the Gaussian random field Ising model at values of the external field strength H where the size of the clusters diverges. For this $T = 0$ system with quenched disorder, the domain Markov property was shown to be satisfied exactly. Together with the conformal invariance of the interfaces deduced from Schramm's formula and the crossing probabilities, it is shown clearly that the spin domain interfaces satisfy $\text{SLE}_{\kappa=6}$, corresponding to pure percolation. This is in contrast to previous observations that some properties of domain walls in the RFIM might be related to tri-critical percolation [19]. The fractal dimension is in perfect agreement with these observations, contrary to the case of the solid-on-solid model studied in Ref. [10], where $\kappa \approx 4$ was found from Schramm's formula, but $d_f \approx 1.25 \neq 1 + \kappa/8$. Studying the SLE map directly, we have shown that the driving function describes Brownian motion. The consistency with SLE carries over to the case of binary random fields, where degeneracies occur. This is in contrast to the observations for the spin glass model, where domain walls appear to be only described by SLE for continuous disorder distributions [6, 7, 30]. The 2D RFIM thus seems to provide a paradigmatic example where SLE is realized in all known aspects in a system with quenched disorder, nourishing the hope for a more systematic treatment of systems with quenched disorder in field theory.

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