

The Physics of Local Optimization in Complex Disordered Systems

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Limited resources motivate decomposing large-scale problems into smaller, “local” subsystems and stitching together the so-found solutions. We explore the physics underlying this approach and discuss the concept of “local hardness”, i.e., complexity from the local solver perspective, in determining the ground states of both P- and NP-hard spin-glasses and related systems. Depending on the model considered, we observe varying scaling behaviors in how errors associated with local predictions decay as a function of the size of the solved subsystem. These errors stem from global critical threshold instabilities, characterized by gapless, avalanche-like excitations that follow scale-invariant size distributions. Away from criticality, local solvers quickly achieve high accuracy, aligning closely with the results of the more computationally intensive global minimization. These findings shed light on how Nature may operate solely through local actions at her disposal.

A prevalent approach to analyzing large-scale problems involves segmenting them into smaller, “local” components which are then solved, followed by assembling the obtained local optimal solutions in a manner akin to a jigsaw puzzle, thereby yielding meaningful results for the original large-scale problem. This *modus operandi* is natural when constrained by limited time and computational resources. It is especially common when one assumes — as often happens in problems with an underlying geometric structure (e.g., images of physical systems [1] or proteins [2]) — that these local solutions provide correct and useful information about the larger system. Such local few-body solvers may accurately capture key features of general many-body systems governed by local Hamiltonians, a result known in certain contexts as *nearsightedness* [3–7]. Here, we explore physical aspects of this approach for determining global ground states (GSs) of both short- and long-range classical spin-glass systems, a known difficult problem [8, 9] using simple local algorithms.

Similar to other complex systems [10–14], the challenge posed by spin-glasses results from the combination of disorder and frustration leading to a complex free-energy landscape [15]. According to recent results these systems generically have very low-lying excited states that are radically different from each other and from the GSs [16–18]. This phenomenon of system-spanning lowest-energy excitations means that optimal (i.e., GS) subsystem solutions might not necessarily match the optimal full system solution. The occurrence of such states is independent of the classification of the corresponding GS problem in the P or NP classes [19]. Indeed, spin-glasses with GSs obtained with polynomial (P) complexity may also exhibit system-spanning lowest-energy excitations [18]. For in-

stance, this is the case for GSs of two-dimensional (2D) Ising spin-glasses on planar graphs, which can be found in polynomial time [20].

Here, we investigate to which extent local (i.e., subsystem) spin-glass solvers approximate global spin-glass GSs. Understanding such features will help uncover the principles that underlie the effectiveness of local solvers. Our numerical analysis (based on exact GS algorithms) concentrates on Ising spin-glasses with Hamiltonian

$$H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j, \quad \sigma_i = \pm 1, \quad (1)$$

with couplings J_{ij} drawn from a Gaussian distribution of mean $\bar{J}$ and unit variance. Unless stated otherwise, we set $\bar{J} = 0$. Here, the N spins are placed on 2D or 3D lattices with open boundaries and nearest-neighbor interactions (Edwards-Anderson (EA) model [21]) or form a Sherrington-Kirkpatrick (SK) [22] model harboring all-to-all interactions (following a rescaling of J_{ij} by $1/\sqrt{N}$ to guarantee a meaningful thermodynamic limit).

Previous related studies attempted to combine multiple local solvers to address global optimization problems [23], fuse local GSs for the construction of global spin-glass instances with planted solutions for benchmarking [24], and develop local spin-glass solvers using gauge-transformation-based deep reinforcement learning [25, 26]. Here, we take on a different perspective and directly investigate to which extent one can predict (parts of) the global GSs based on optimization of local subsystems. To reduce this problem to its bare essentials and minimize boundary effects, we study pairs of neighboring spins (bonds) of the model and introduce a “local single bond solver” (LSBS) that examines these neighboring spins within a local subsystem. We first focus on

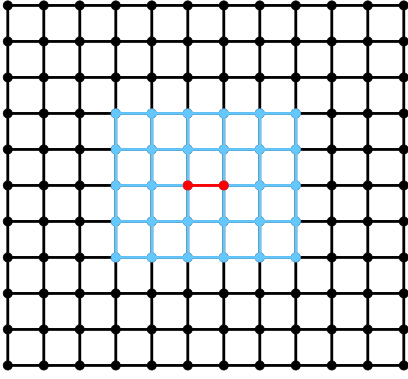


FIG. 1: Computational setup for determining the relative spin orientation $\sigma_{i_0}\sigma_{j_0}$ across the central bond (shown in red). We set the lattice constant to unity. The correct value of $\sigma_{i_0}\sigma_{j_0}$ is that within the GS of the entire system (comprised of black, light blue, and red bonds). The LSBS computes $\sigma_{i_0}\sigma_{j_0}$ within the GS of the local subsystem (light blue and red bonds).

the GS products $\sigma_{i_0}\sigma_{j_0}$ for bonds $\langle i_0j_0 \rangle$ which, in the simplest case, are located at the center of the system, cf. Fig. 1.

We are interested to see whether the value of this spin product in the global GS matches that obtained by the LSBS for a subsystem centered on this single bond. Recent studies [27] examined correlations between adjacent spins at the system center after resampling its “shell.” By contrast, here we examine whether this product as determined by (i) GS computations for small *subsystems* containing this bond and using open boundaries agrees with that in (ii) the *global system* GS. As we shall see, system-spanning correlated changes to spin-glass GSs appear only at special critical interaction strengths [18]. When the spin-glass couplings are far removed from their critical threshold values, at which global avalanche-like changes occur, the GSs are robust to small variations of the coupling constants. This robustness enables the LSBS to become accurate whenever coupling constants are far from their critical values. We find that for large lattice systems, the LSBS error rate becomes independent of the global system size and exhibits a power-law decay with subsystem size. More generally, several other *distance-dependent* quantities also show *power-law* behaviors (see also Ref.[28], S7). By contrast, the dependence of the LSBS error rate on the deviations of the coupling from its critical value assumes a stretched exponential form.

To systematically study the relation of local and global solutions in spin systems, we rely on exact (combinatorial) GS algorithms, using a minimum-weight perfect matching (MWPM) method [29] based on Blossom V [30] for the polynomial 2D problem and the general-purpose Gurobi software [31] for 3D systems and the SK model.

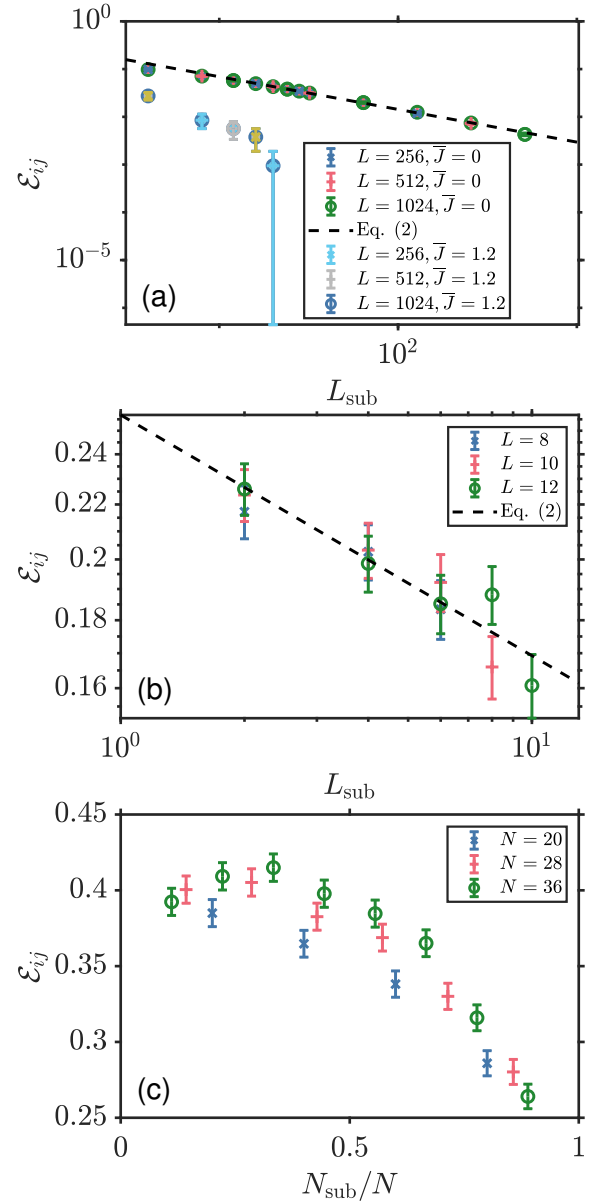


FIG. 2: The disorder-averaged LSBS error rate $\mathcal{E}_{ij}$ as a function of the subsystem size L_{sub} (resp. N_{sub}) for (a) the square-lattice (2D) EA model, (b) the cubic lattice (3D) EA model, and (c) the SK model. For each plot, the results for varying system sizes L (N) are shown. For (a), both cases $\bar{J} = 0$ and $\bar{J} = 1.2$ are shown, whereas in (b) and (c) $\bar{J} = 0$. Note the nearly perfect *collapse* of different system-size data indicating that the error of the LSBS becomes asymptotically *independent* of system size. For the lattice systems in (a)–(b) and $\bar{J} = 0$, the error rate is fit by Eq. (2), cf. the straight dashed lines. Unlike these lattice systems, the all-to-all SK model of (c) is devoid of geometric locality. Here, N_{sub} spins were randomly chosen to comprise the subsystem.

Figure 1 schematically depicts the setup: we computed

d	$\ell_{\mathcal{E}}$	κ	k_J	α
2D	0.20(1)	0.685(8)	10.9(5)	1.35(5)
3D	0.0006(10)	0.18(3)	0.47(3)	1.4(1)

d	β	ℓ_c	a_c
2D	0.198(8)	1.01(3)	1.26(1)
3D	1	1.40(9)	1.3(1)

TABLE I: Parameters of the fits of the functional forms of Eqs. (2), (3), and (4) to our data. In 2D, $\chi^2/\text{d.o.f}$ is found to be 0.821, 0.477, and 1.067 for Eqs. (2), (3), (4), respectively. In 3D, these values are 1.265, 0.794, 0.656. Note that for the fit of the form (3) in 3D, we fixed $\beta = 1$. For the 2D systems, the form (2) was fitted with a cutoff $L_{\text{sub}} \geq 16$ in order to minimize finite-size effects.

GSs for the entire (N -spin) systems as well as for subsystems with N_{sub} lattice sites centered on the considered bond $\langle i, j \rangle$ (corresponding to the LSBS), comparing the predicted values of the product $\sigma_{i_0}\sigma_{j_0}$ (open boundary conditions are applied in both cases). If the results agree, the bond product was predicted correctly (with zero error). For $L \times L$ square lattices of sizes $L = 256, 512, 1024$, we examined $L_{\text{sub}} \times L_{\text{sub}}$ subsystems with $4 \leq L_{\text{sub}} \leq 512$, using $\sim 60,000$ disorder realizations. For cubic lattices of side lengths $L = 10$ and 12 , the $L_{\text{sub}} \times L_{\text{sub}} \times L_{\text{sub}}$ subsystems were of size $4 \leq L_{\text{sub}} \leq 10$, and we used $\sim 2,000$ realizations. From these calculations we estimate the LSBS error rate $\mathcal{E}$. Figure 2 shows the dependence of $\mathcal{E}$ on the subsystem size for several cases. For the $\bar{J} = 0$ lattice systems we find that, although the 2D and 3D spin-glass GS problems belong to different complexity classes (P and NP, respectively), in both cases the error rate is well described by a power-law decay in the subsystem size [32] [33], viz.

$$\mathcal{E}_{ij} \equiv \frac{1 - [\sigma_i \sigma_j \sigma_i^{\text{sub}} \sigma_j^{\text{sub}}]}{2} \sim (\ell_{\mathcal{E}}/L_{\text{sub}})^{\kappa}. \quad (2)$$

Here, $[\sigma_i \sigma_j \sigma_i^{\text{sub}} \sigma_j^{\text{sub}}]$ denotes the pair-overlap correlation function, where $[\dots]$ represents the disorder average, and $\ell_{\mathcal{E}}$ is an effective length scale. Importantly, this algebraic decay is *independent of the linear system size L* , cf. the data collapse in Figs. 2(a) and (b). Comparing the respective exponents κ that are collected in Table I, we see that errors in the 2D systems drop more rapidly than in the (harder) 3D systems. When expressed in terms of the total volume $N_{\text{sub}} = L_{\text{sub}}^d$, where d is the lattice dimension, this sharp change in the algebraic decays of errors between the 2D and 3D cases becomes yet more acute.

While the precise power-law decay differs between 2D and 3D, this variability is much less dramatic than maybe expected from the P vs. NP contrast in computational complexity between these cases. This observation suggests a “local hardness” computational complexity de-

scriptor beyond the P-NP classification [19] (and related categories [34]) shedding further light on the intrinsic difficulty of spin-glass GS computations. By *local hardness*, we allude to *how large the subsystem* on which computations are done must be so that, when averaged over many instances, the local solver provides a correct answer up to a fixed *error rate*. Arguably, the local complexity is a relevant descriptor for physical local measurements (the microscopic state of the full many-body system cannot, in general, be probed).

For the case of non-zero average coupling $\bar{J}$, we find the same power-law decay as long as we remain in the (zero-temperature) spin-glass phase, thus suggesting some degree of universality in the behavior of local hardness. As soon as one enters the ferromagnetic phase, however, there is a much faster, exponential decay of the error rate, cf. the data for $\bar{J} = 1.2$ in Fig. 2(a), Fig. 7 (End Matter) and further details in Ref. [28], Sec. S5. Similarly, we find identical 2D-type scaling and exponents for planar anti-ferromagnets having faint random perturbations of their couplings from an otherwise constant value (Ref. [28], Sec. S6). For the all-to-all SK model, on the other hand, the data in Fig. 2(c) reveal that the local hardness *does* depend on the total system size N and not only on the N_{sub} randomly chosen spins forming the “local” subsystem, thus indicating a higher degree of local hardness of the SK model. This is consistent with the fact that there is *no local geometry* in the SK system, and that it consequently features a more complex free-energy landscape with infinitely many thermodynamic states [35]. As a result, one needs to scale N_{sub} proportional to N to achieve low error rate for the LSBS in the SK model.

We next turn to our central endeavor — that of understanding the physics underlying local solvers. Our approach relies on the concept of “critical thresholds” $J_{c,ij}$ [18, 36], defined as follows. Across any bond, the spin product $\sigma_i \sigma_j$ is either $+1$ or -1 . Consider now continuously varying J_{ij} from $-\infty$ to $+\infty$ leaving the strengths of all other bonds unchanged. As we have shown elsewhere [18], there is a *unique* value $J_{ij} = J_{c,ij}$ where the relative orientation of σ_i and σ_j changes from $\sigma_i = -\sigma_j$ ($J_{ij} < J_{c,ij}$) to $\sigma_i = \sigma_j$ ($J_{ij} > J_{c,ij}$) and, hence, two GSs become degenerate. A similar definition applies for the subsystem — changing the coupling J_{ij} beyond $J_{c,ij}^{\text{sub}}$ leads to a change of the subsystem GS. For the LSBS we have the following important observation: If $J_{c,ij}^{\text{sub}}$ and $J_{c,ij}$ are either (i) both smaller or (ii) both larger than J_{ij} , then the LSBS (determining the value of $\sigma_i \sigma_j$ as calculated within the smaller subsystem GS) will yield the correct $\sigma_i \sigma_j$ value (i.e., that computed within the global system GS). The opposite case of J_{ij} being sandwiched between $J_{c,ij}$ and $J_{c,ij}^{\text{sub}}$, leading to a failure of LSBS, is more likely to occur if J_{ij} is close to the critical thresholds, cf. Fig. 3. Thus, the separation $|J_{c,ij}^{\text{sub}} - J_{ij}|$ can serve as a predictor of LSBS reliability [37]. When $J_{c,ij}^{\text{sub}}$

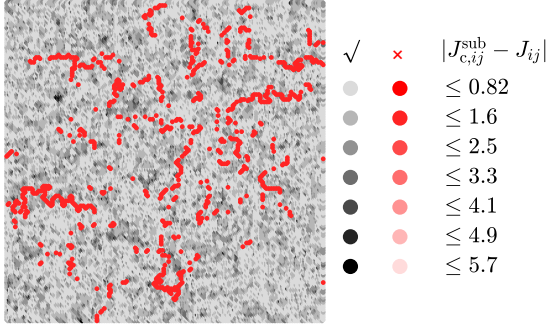


FIG. 3: LSBS solutions for general nearest-neighbor bonds $\langle ij \rangle$ in an $L = 128$ system. An $L_{\text{sub}} = 40$ subsystem is chosen to be centered about any such bond $\langle ij \rangle$; near the boundaries, the subsystem becomes correspondingly smaller. $J_{c,ij}^{\text{sub}}$ denotes the critical threshold of bond $\langle ij \rangle$. ✗ denotes an error wherein the local solver does not match the global solution, while ✓ refers to correct LSBS predictions. Errors arise more readily for smaller $|J_{c,ij}^{\text{sub}} - J_{ij}|$.

and J_{ij} are far apart, then $J_{c,ij}^{\text{sub}}$ and $J_{c,ij}$ may indeed easily be both larger or both smaller than J_{ij} . Conversely, when $J_{c,ij}^{\text{sub}}$ and J_{ij} are close, slight miscalculations can lead to $J_{c,ij}^{\text{sub}}$ and $J_{c,ij}$ to appear on opposite sides of J_{ij} , thus causing the LSBS to yield incorrect $\sigma_i \sigma_j$ values.

At the critical threshold J_c , the GS responds sensitively to infinitesimal single-bond changes. Similar to avalanches in the random-field Ising model [38], neural networks [39], systems featuring Highly Optimized Tolerance [40], and other problems [41], the GS spins that are flipped as a result of such changes form fractal boundary Zero Energy Droplets (ZEDs) of sizes conforming to power-law distributions [18]. These distributions may be tied to complex critical spatial correlations (Ref. [28], Sec. S11). Configurations associated with general excitations may be constructed as composites of the (single bond) ZEDs [18]. These excitations are, by construction, of vanishing energy. Right at the thresholds, owing to the sudden sensitivity of the global *fundamentally discrete* (i.e., Ising) spin-glass GS problem to infinitesimal local changes in the continuous bond couplings, the LSBS achieves low accuracy. However, when the coupling constants are sufficiently removed from these (measure zero) critical thresholds, the local solver becomes progressively more accurate.

Figure 3 illustrates the *sensitivity* $|J_{c,ij}^{\text{sub}} - J_{ij}|$ of each bond of a given $L = 128$ square-lattice sample, and how it relates to the error rate of the LSBS. As is seen here (and in Fig. 4, End Matter), the LSBS error rate decays with increasing $|J_{c,ij}^{\text{sub}} - J_{ij}|$. We find this decay to be described well by a stretched-exponential form

$$\mathcal{E}_{ij} \sim k_J e^{-\alpha |J_{c,ij}^{\text{sub}} - J_{ij}|^\beta}. \quad (3)$$

The parameters for $L = 1024$, $L_{\text{sub}} = 8$ square and $L =$

12, $L_{\text{sub}} = 4$ cubic lattice systems appear in Table I. The modulus $|J_{c,ij}^{\text{sub}} - J_{ij}|$ is equal to twice the excitation energy ΔE , cf. Eq. (S1) in Ref. [28], Sec. S1. Thus the accuracy of the LSBS for a certain bond is determined by the respective excitation energy of the bond in this subsystem.

We further tuned the central bonds of $L = 128$ square and $L = 12$ cubic lattices to their critical values and then computed the respective critical threshold changes $(\Delta J_{c,ij})$ of all other bonds. We find that the average difference $|\Delta J_{c,ij}|$ decays algebraically with distance from the modified central bond (see Fig. 5, End Matter),

$$[|\Delta J_{c,ij}|] \sim (\ell_c/r)^{a_c}. \quad (4)$$

In 2D and 3D systems, both the length ℓ_c and the exponent a_c are of order unity, the fit parameters are provided in Table I. For sufficiently large r , the central bond coupling negligibly impacts the $J_{c,ij}$ of distant bonds. In other words, the critical coupling of a given bond is almost completely determined only by the couplings on bonds that are close to it. This exemplifies the “locality of the critical threshold $J_{c,ij}$.” Since the ZED volume distribution decays algebraically [18], for sufficiently large distances r between the tuned and observed bond, the probability of ZED excitations connecting these bonds becomes small. When resampling the system boundary, $[|\Delta J_{c,ij}|]$ decays similar to Eq. (4), see Ref. [28], Sec. S7. Thus, generating a subsystem by removing peripheral bonds is of limited impact on the critical coupling of the central bond thus explaining the proximity of $J_{c,ij}^{\text{sub}}$ and $J_{c,ij}$.

In summary, using *exact GS algorithms*, we study the effectiveness of local subsystem solvers and relate it to critical threshold ZED physics [18] of system-spanning avalanche-like vanishing-energy excitations. In short-range spin-glasses, away from these special critical thresholds, the response of the system to local perturbations to the GS problem decays rapidly. The difference between local minimization and its full global counterpart becomes significant primarily near the critical coupling threshold values. Away from critical threshold values, local GS solvers achieve increased accuracy. Importantly, the associated error rate of the local solvers becomes *independent of the system size* [see Fig. 2 (a)] for large systems and decreases rapidly with the subsystem size. Local solvers can thus become *increasingly effective* for large L where a local solution of a subsystem that is a very small fraction $(L_{\text{sub}}/L)^d$ of the large system may correctly predict local properties of the large system with negligible error. Simple functions link the average error rate of local solvers to their (subsystem) size as well as to deviations of the coupling constants from their critical threshold value, and changes of the critical coupling of one bond as a result of tuning another distant bond to its critical threshold value. Our analysis reveals that various length-dependent quantities appear

to display algebraic scaling just like the scale-free distributions of avalanche size [18], while for ferromagnetic phases we observe an exponential decay of errors with subsystem size (see also End Matter and [28], Sec. S5). These behaviors may rationalize the success of more sophisticated local solvers in spin-glasses [24–26] and other systems [1, 2], and might be broadly applicable in systems not constrained by the “Overlap Gap Property” [42]. For the all-to-all SK model, on the other hand, the error rates always depend on the size of the full system. These differences lead us to conjecture a classification of possible levels of local hardness (Ref. [28], Sec. S4). Local solvers are found to be efficient for the considered lattices irrespective of whether the global problems are of P (2D) or NP (3D) complexity. The occurrence of rare events in the form of bonds attaining their critical couplings explains why local solvers cannot in general guarantee to find exact GSs for the full systems.

We briefly speculate on broad implications and extensions of our results regarding the effectiveness of local optimization. Efficient and often highly accurate local minimization is of immense utility in replacing more taxing global (“batch”) gradient descent methods by inherently fast sequential local stochastic gradient descent-type approaches across numerous problems including high-dimensional/large-data machine learning type tasks in many-body physics, materials science, community detection, and other domains [43–48]. Local plastic variants of the backpropagation algorithm used in machine learning might emulate efficient biological neural computing [49, 50]. Other notable aspects of local minimization and optimized structure in artificial and real neural networks have been further investigated [51, 52]. Locality is of paramount importance in numerous problems in many-body physics [3–7] where the degrees of freedom may also be *continuous*. Replacing Eq. (1) by $H = -\sum_{ij} J_{ij} s_i s_j + u \sum_i (s_i^2 - 1)^2$ with $u \gg 1$ (Eq. (1) corresponds to $u \rightarrow \infty$) for continuous $s_i \in \mathbb{R}$ suggests that the phenomena that we found might extend to continuous spin systems of sufficiently large quartic interaction strength u . In a specific continuous spin variant, that of the spherical model [53, 54], the locality that we described here is altered (Ref. [28], Sec. S9). In future work, we will also further explore viable relations to previously unreported criticality (Ref. [28], Sec. S11). Ultimately, the physics of local optimization may underlie Nature’s enigmatic efficiency, shaping computation across diverse realms.

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END MATTER

Here, we provide numerical evidence for Eqs. (3) and (4), the existence of a transition in the LSBS accuracy as the distribution of couplings is shifted (so as make the system more ferromagnetic and easier to solve), and we discuss beyond nearest-neighbor correlations found by the LSBS.

Error rates for deviations from the critical threshold

Illustrating the applicability of the analysis in the main text, Fig. 4 demonstrates that the LSBS error indeed decays with increasing $|J_{c,ij}^{\text{sub}} - J_{ij}|$ in a manner that adheres to the stretched exponential drop of Eq. (3).

Distance dependence of critical threshold values

As discussed in the main text, the critical threshold of a given bond is largely determined only by the couplings of other bonds that are very close to it. When we tuned the central bonds of $L = 128$ square and $L = 12$ cubic lattices to their critical values, the respective critical threshold changes ($\Delta J_{c,ij}$) of all other bonds at a dis-

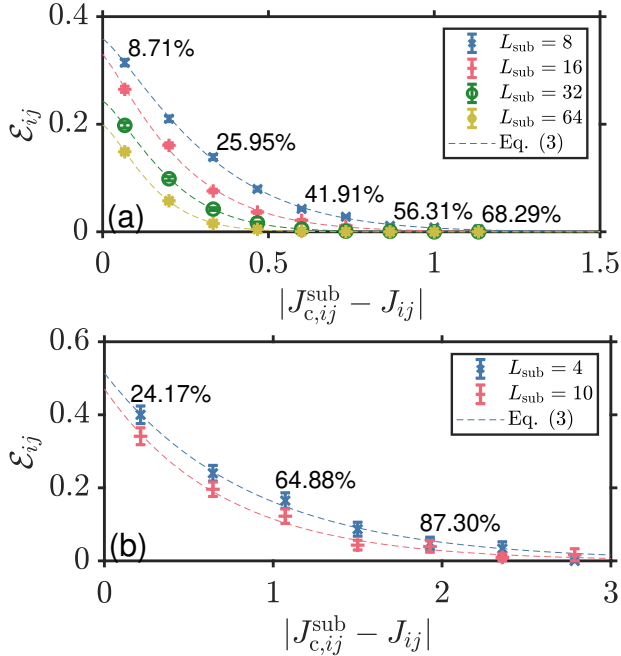


FIG. 4: Error rate $\mathcal{E}_{ij}$ as a function of $|J_{c,ij}^{\text{sub}} - J_{ij}|$ for (a) a 2D (square lattice), $L = 1024$ system with $L_{\text{sub}} = 8, 16, 32, 64$ and (b) a 3D (cubic) system of $L = 12$ with $L_{\text{sub}} = 4, 10$. Point percentages denote cumulative probabilities (i.e., fraction of instances) for values of $|J_{c,ij}^{\text{sub}} - J_{ij}|$ lower than their abscissa.

tance r away conformed to Eq. (4) with ℓ_J of the order of the lattice constant, see Table I (with the fit of Eq. (4) becoming an upper bound for the largest distances in the 2D system). This is illustrated in Fig. 5. In Fig. 6, we further provide a calculated real-space visualization of such a typical rapid drop of the change in the criti-

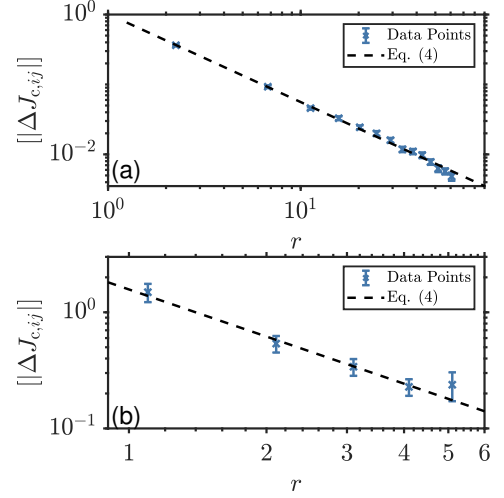


FIG. 5: Average change of critical threshold values ($|\Delta J_c|$) of bonds at a distance of r along a Cartesian direction from $-\infty$ to $+\infty$ in (a) 2D ($L = 128$) and (b) 3D ($L = 12$) systems. Dashed: Eq. (4) with parameters in Table I. As is more evident in 2D, when $r = \mathcal{O}(L)$, the average $|\Delta J_c|$ is bounded from above by these ultra-local forms with the fitted ℓ_c being of the order of the lattice constant.

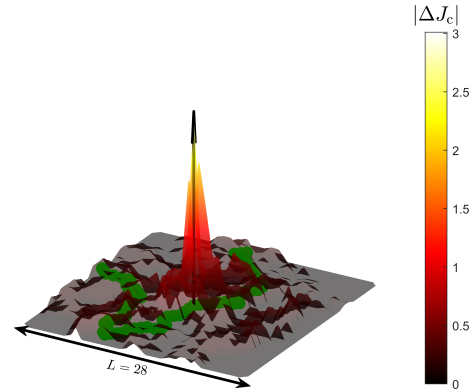


FIG. 6: Typical real space realization of the data in Fig. 5. On an $L = 28$ square lattice, altering the central bond coupling $J_{i_0 j_0}$ from $-\infty$ to $+\infty$ generally induces the change of critical thresholds $J_{c,ij}$'s of all other bonds in the system. The surface color map visualizes the amplitude of $J_{c,ij}$'s change for a typical realization of the system. The green contour represents the flipped bonds between the GSs before and after J_{ij} crosses its own $J_{c,ij}$ as it is monotonically varied.

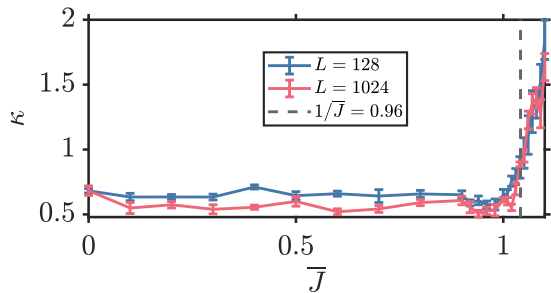


FIG. 7: For an asymmetric bond distribution, as $\bar{J}$ increases the system transitions from a spin-glass phase to a disordered ferromagnetic phase [55]. We observe how the fitting coefficient κ changes as we adjust $\bar{J}$. A larger value of κ implies that the error rate of the subsystem solver decays faster, with a crossover to an exponential decay beyond the threshold $1/\bar{J} = 0.96$, where the system enters the ferromagnetic phase [55].

cal coupling away from the origin. We further observe a smaller effect on distant bonds for smaller ZEDs.

Shifted Gaussian Distribution of Couplings

With the exception of Fig. 2 (a), in the main text, we largely focused on a symmetric Gaussian distribution of coupling constants (one centered about $\bar{J} = 0$). As $\bar{J}$ becomes larger, the system becomes less frustrated; in the trivial large $\bar{J}$ limit, the GS becomes ferromagnetic. Naturally, everything else being kept fixed, for large $\bar{J}$, the error $\mathcal{E}_{ij}$ of Eq. (2) must decay to zero more quickly. In Fig. 7, we illustrate how such a transition is manifest when $\bar{J}$ is of order unity with the exponent κ in Eq. (2) increasing rapidly (Ref. [28], Sec. S5).

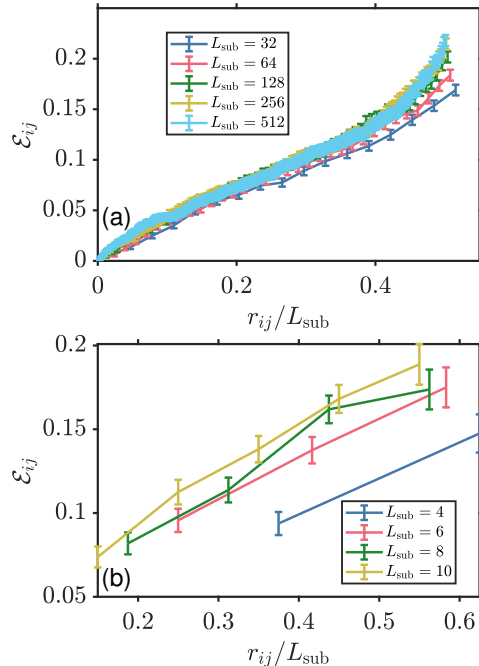


FIG. 8: LSBS error rate for GS bond products $\sigma_i \sigma_j$ when sites i, j are a distance r_{ij} apart along a Cartesian axis. Results for (a) 2D and (b) 3D systems. The abscissa is normalized by the subsystem size, r_{ij}/L_{sub} . As L_{sub} increases, $\mathcal{E}_{ij}$ gradually collapses onto a single curve (as for the $r_{ij} = 1$ case of the main text).

Extension of the LSBS to distant spins

In the main text, we examined the accuracy of local solvers in determining nearest-neighbor spin products (or correlations) within the GS. In Fig. 8, we illustrate the result of our approach when the spins σ_i and σ_j are no longer nearest neighbors but are rather a distance r_{ij} away from each other along the Cartesian axes. Again, we compute the spin product within the GS of a local subsystem that contains these two spins. As may be expected, as the distance r_{ij} becomes larger relative to the subsystem size the error rate of the local solver increases.

Supplemental Material for: “The Physics of Local Optimization in Complex Disordered Systems”

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S1. COMPUTATION OF CRITICAL THRESHOLD COUPLING

To determine the critical threshold $J_{c,ij}$, we first compute the GS of the system and extract $\sigma_i\sigma_j$, the value of this nearest-neighbor spin product in the GS with the original coupling on the link $\langle ij \rangle$ which we here explicitly denote by $\underline{J}_{ij}$. Then, we modify the coupling J_{ij} so that it has a sign opposite to that of $\underline{\sigma_i\sigma_j}$ with $|J_{ij}|$ sufficiently large so as to force the product of spins associated with this bond ($\sigma_i\sigma_j$) to change its sign in the new resulting GS. Next, this GS for the now modified coupling is determined; this new state constitutes an excited state (of energy $\Delta E > 0$) for the original system (in which J_{ij} is not changed). Once the energy ΔE is numerically computed the critical coupling may then be determined via the equality

$$J_{c,ij} = \underline{J}_{ij} - \underline{\sigma_i\sigma_j}\Delta E/2. \quad (S1)$$

On general lattices and graphs, the probability density of critical couplings depends mainly on the (inherently *local*) coordination number [1]. This suggests that, on average, general functions f depending on $J_{c,ij}$ may be local (this includes the Heaviside function $f = \Theta(J_{ij} - J_{c,ij})$ which provides the aforementioned sign of the bond $\langle ij \rangle$ (i.e., the product $\sigma_i\sigma_j$) that the LSBS aims to find). Applied to any subsystem containing the link $\langle ij \rangle$, we may similarly compute the *subsystem critical coupling* $J_{c,ij}^{\text{sub}}$.

S2. CRITICAL AVALANCHES AT TRANSITIONS BETWEEN GLOBAL GSS

Ref. [1] examined, in disparate lattices, the transitions between different global solutions when a single nearest neighbor coupling traversed its critical value $J_{c,ij}$ to universally find scale invariant power law distributions for ZED volume and area [1]. In all studied 2D and 3D lattices, critical system spanning avalanches appear at $J_{c,ij}$ in which a divergent number of spins change their values. These avalanches are characterized by nontrivial

exponents that (within obtainable numerical accuracy) are largely determined only by the spatial dimensionality and insensitive to specific lattice details. As explained in the main text, the rarity of these transitions — the fact that they only occur at special $J_{c,ij}$ values — underlies the accuracy of the local optimization problem solved by LSBS (as compared to the exact full system GS). In Fig. S1, we provide an example of a typical critical avalanche that occurs at $J_{c,ij}$. Various details of these avalanches (which constitute zero energy droplet (ZED) excitations at $J_{c,ij}$) are given in Ref. [1].

S3. MORE ON A CORE CONCEPT UNDERLYING THE LSBS

As discussed in the main text (and illustrated in Fig. 3), a central theme lying at the heart of the LSBS concerns the proximity of the critical threshold value of the same bond (ij) in subsystem and in the full system. If the full system (J_c) and subsystem (J_c^{sub}) critical values for the bond (ij) are close to each other then the local solver may still remain accurate for couplings J_{ij} that are not far the critical threshold value in the subsystem J_c^{sub} . Specifically, if the coupling $J_{ij} > \max(J_c^{\text{sub}}, J_c)$ or $J_{ij} < \min(J_c^{\text{sub}}, J_c)$ then the LSBS will provide the exact answer. The only “dangerous” situation in which the LSBS will provide the incorrect result is when the coupling for the bond (ij) will lie between its global critical $J_{c,ij}$ value and its subsystem critical threshold value J_c^{sub} . If the latter threshold values are very close to each other then the probability this “dangerous” situation will occur can become exceedingly low. The graphic of Fig. S2 further depicts this key notion. As illustrated in Fig. 3 of the main text, as the coupling J_{ij} becomes far removed from the local critical coupling J_c^{sub} for the bond (ij), the LSBS indeed achieves higher accuracy.

S4. LEVELS OF LOCAL HARDNESS

Before proceeding with the enumeration of viable formal levels of local hardness, we must underscore that the

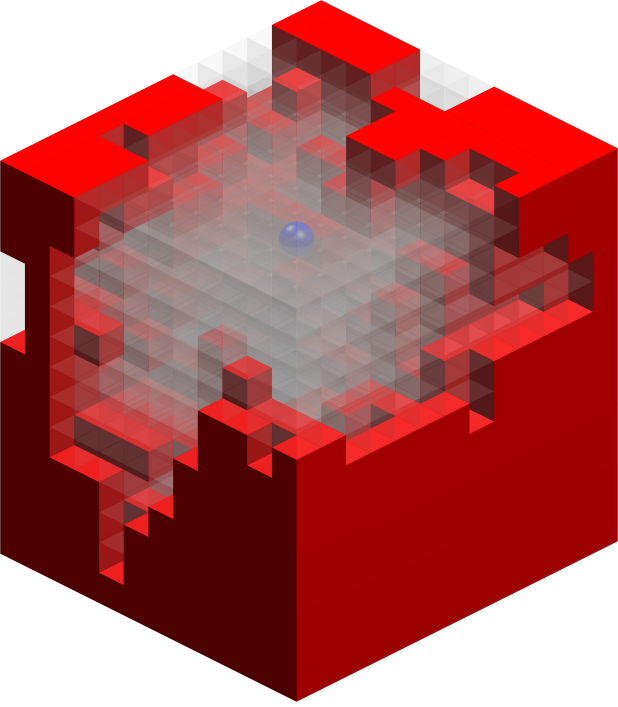


FIG. S1: A computed critical avalanche associated with a transition (see text) between optimal global solutions—different GSs of the EA model. When the local spin-glass solution provided by the LSBS (Fig. 1) for the central bond $\sigma_{i_0}\sigma_{j_0}$ (blue sphere) differs from the global one, the flipped spins form bubbles (above “cloud” surrounding the central bond) that occur at all scales (governed by critical power law distributions) [1]. Shown here is a typical critical avalanche for an $L = 12$ cubic lattice.

conventional P/NP classification relates to exact solutions of the full global problem (a vanishing error rate)—not that for a finite error rate as discussed in the current work which may naturally arise when only solving for a local fragment of the problem.

We next couch our spin-glass GS problem in a far broader context and ask how hard it generally may be for local solvers to solve global optimization problems. When examining systems (problems) composed of a large number of spins (or elements) $N \gg 1$, different levels of local hardness are conceivable:

- I. There exists an N^* that is independent of the entire system size (total number of spins) N , such that as long as the size of the subsystem is larger than N^* , the error rate of the subsystem GS solver is strictly always 0.
- II. For any maximally tolerable error rate ϵ , there exists a subsystem size $N^*(\epsilon) < N$ that does not depend on the entire system size N , such that solving for the GSs in subsystems $N_{\text{sub}} \geq N^*(\epsilon)$ yields an error rate that is $\leq \epsilon$.

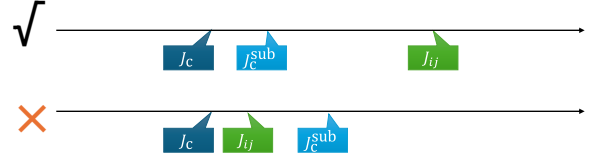


FIG. S2: A schematic of the relative sizes of the (i) the bond coupling constant J_{ij} of a bond $\langle ij \rangle$ (green), (ii) the exact coupling constant $J_{c,ij}$ (navy blue) of this bond within the entire system and (iii) the critical threshold $J_{c,ij}^{\text{sub}}$ (light blue) of this bond as computed within a local patch or subsystem that includes this bond at its center. The exact GS spin product $\sigma_i\sigma_j = \pm 1$ when, respectively, the coupling $J_{ij} \gtrless J_{c,ij}$. The value of $\sigma_i\sigma_j$ as determined by the approximate local (subsystem) bond solver coincides with the exact full system GS answer when both $(J_{c,ij}^{\text{sub}} - J_{ij})$ and $(J_{c,ij} - J_{ij})$ are of the same sign (✓ at top panel). When the signs of $(J_{c,ij}^{\text{sub}} - J_{ij})$ and $(J_{c,ij} - J_{ij})$ are opposite, the local bond solver yields an incorrect prediction (✗ at bottom panel). As we will later detail (Eq. (S3) and Fig. S8), we indeed find that with increasing subsystem size, the deviation between the global and subsystem critical threshold value decreases, thus further consistently explaining the increasing accuracy of the LSBS with subsystem size.

- III. For any maximally tolerable error rate ϵ , there exists a subsystem size $N^*(\epsilon, N) < N$ that depends on the entire system size N (and may diverge as $N \rightarrow \infty$), such that solving this subsystem can yield an error rate $\leq \epsilon$.
- IV. Even if the subsystem size is made arbitrarily large so long as it is smaller than that of the full system (i.e., $N_{\text{sub}} < N$), it is impossible to guarantee that the error obtained when comparing the exact GS of the subsystem with that of the full system can be made lower than a finite error rate ϵ_* .

The diametrically opposite extremes of Type I and Type IV hardness are known from numerous simple contexts. For instance,

- Most conventional textbook type systems (e.g., the classical ferromagnet or antiferromagnet) correspond to type I systems. (In the classical ferromagnet and antiferromagnet, small finite patch subsystem (respectively, uniform or staggered magnetization) GSs capture the exact GSs of the full system.)
- Type IV hardness may, e.g., appear for problems exhibiting the Overlap Gap Property [2] as well as bona fide random systems. Indeed, if we were to guess the product of N randomly sampled numbers ± 1 that are chosen with equal probability, the estimates on products of numbers in smaller subsystems of any size $N_{\text{sub}} < N$ would always have a 50% error rate.

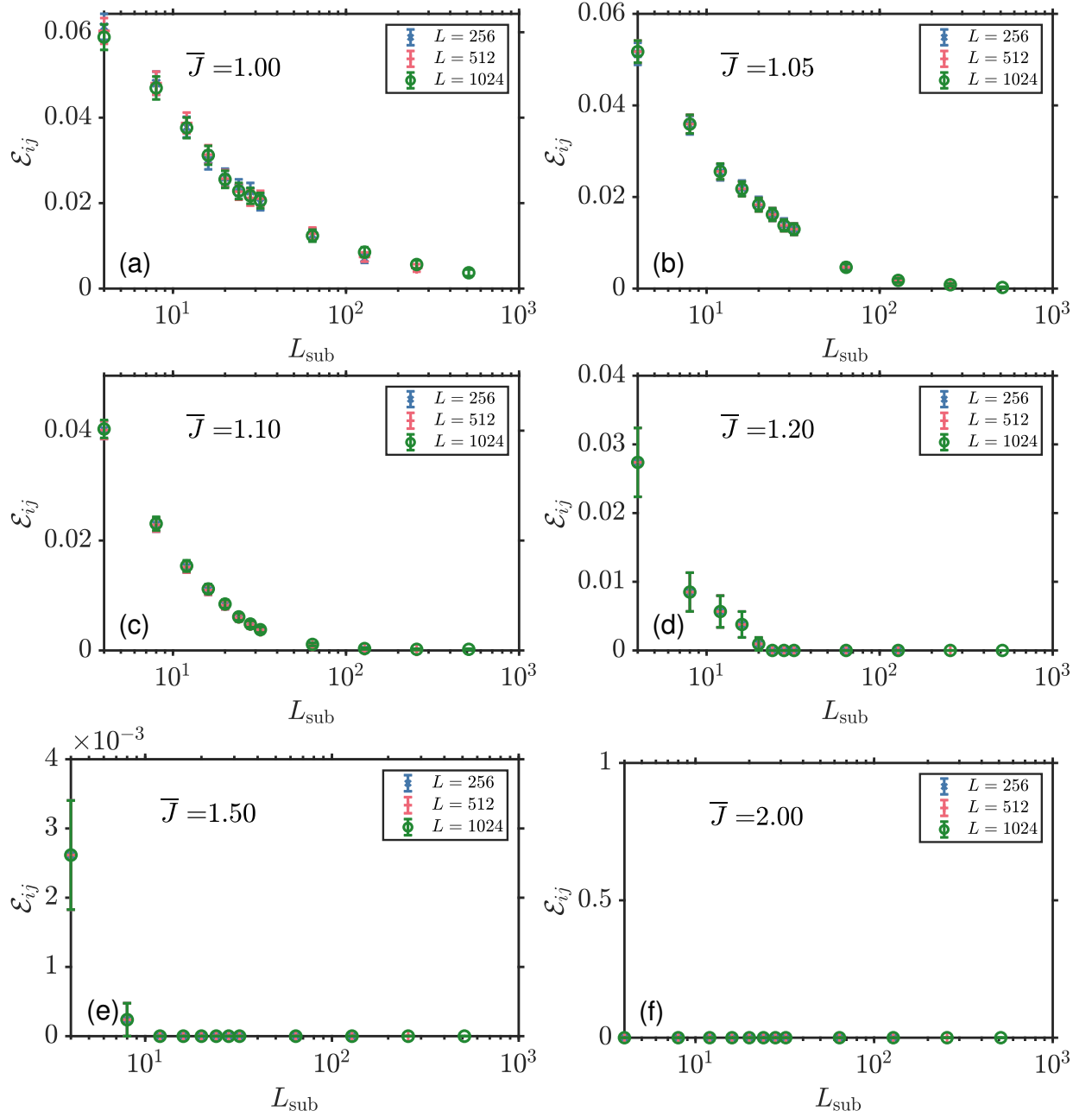


FIG. S3: (a)-(f): The performance of the LSBS on 2D spin-glasses having a shifted Gaussian distribution of their couplings (mean $\bar{J} \neq 0$). In panels (a)-(f), the respective values are $\bar{J} = 1.0, 1.05, 1.10, 1.20, 1.50$, and 2.0 . The setting is similar to that of Fig. 2 of the main text. As $\bar{J}$ is elevated for the shown values, the error rate quickly tends to zero with increasing subsystem length L_{sub} .

The intermediate levels of hardness that we find suggest an underlying richer fine structure of the local complexity. Inasmuch as finite-size numerical results can be extrapolated to the thermodynamic limit, our calculations indicate that spin-glasses (and thus other general complex problems) straddle and are connected (when control parameters are further varied) to the above extreme limits of I and IV. In particular,

- Our computations (see Fig. 2) suggest that the local

hardness of computing both 2D and 3D spin-glass GSs is that of level II.

- The 2D spin glass having its couplings sampled from a ‘shifted’ univariate Gaussian distribution for the coupling constants having a mean of $\bar{J} = 1.2$ has level I hardness (see Sec. S5 for the other shift values). Interestingly, in the latter case, we found a hardness transition point $\bar{J} \simeq 1.1$, which is marginally consistent with $1/r_c = 1.04(1)$, as the ferromagnet-to-spin-glass transi-

tion value of $\bar{J}$ reported in the literature [3].

- The local hardness associated with computing the GSs for the fully connected SK model [4] is (at least) that of level III.

S5. LSBS EXAMPLES FOR SHIFTED GAUSSIAN DISTRIBUTIONS

In the End Matter (Fig. 7), we illustrated the existence of a transition in the error rate as the mean of the Gaussian distribution of couplings was shifted by an amount $\bar{J}$. Here, we provide schematics for the results obtained for different $\bar{J}$. Fig. S3 shows sample plots of the associated rapid drop of the error rate in subsystem size with increasing $\bar{J}$ from which such values of κ are deduced. In the taxonomy that we introduced above when discussing the levels of local hardness, as the Gaussian distribution of the coupling constants J_{ij} is progressively biased towards larger $\bar{J}$ values, the performance of the LSBS improves. These changes (see also Fig. 7) are suggestive of a transition from level II to level I hardness at $\bar{J} \approx 1$.

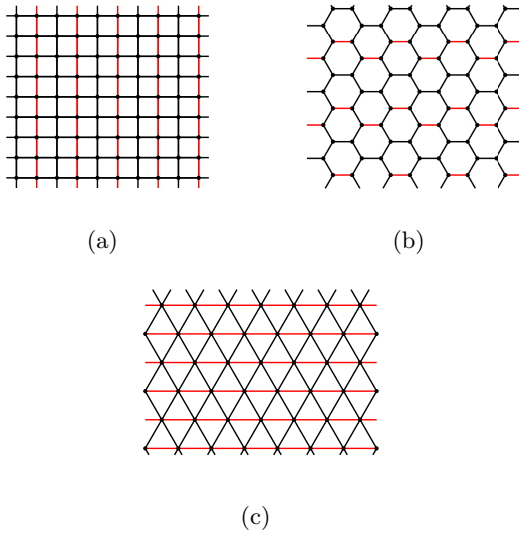


FIG. S4: Specific nearest-neighbor coupling realizations in our studied fully-frustrated square (a), honeycomb (b), and triangular (c) lattice spin systems. The red and black lines respectively denote equal strength bonds with ferromagnetic and antiferromagnetic couplings. The product of the coupling constants around any minimal (respectively, square, hexagonal, or triangular) plaquette is negative. As described in the text, these bonds were weakly perturbed to lift an otherwise exponential GS degeneracy which would render the problem determining the sign of the central bond in the GS meaningless.

S6. LSBS ON 2D FULLY-FRUSTRATED SYSTEMS

A natural question concerns the extension of local solvability that we studied in the main text for spin-glass systems to several fully-frustrated systems. Specifically, towards that end, we apply the LSBS to fully-frustrated 2D spin systems on the (i) square, (ii) triangular, and (iii) honeycomb lattices. In these fully-frustrated systems, within every minimal (respectively, single square, triangular and hexagonal) plaquette, at least one bond cannot be satisfied. In Fig. S4, we provide an illustration of these systems. The specific setup of these systems is as follows: as seen in the figure, some bonds in the system are set to be ferromagnetic, while others are antiferromagnetic with the product of all bonds around a plaquette being negative. In the absence of disorder, such systems may have an exponentially large GS degeneracy (in the full system size N) in which local bonds are not uniquely determined (as in, e.g., Wannier classical result [5] for the triangular lattice Ising antiferromagnet and Villain's demonstration of the exponential degeneracy of the fully frustrated Ising square lattice model [6] invoking Fisher's asymptotic result for dimer coverings [7]). This renders a comparison between the LSBS to the global GSs ill-posed since both single bond ($\sigma_i \sigma_j$) values are possible in a global GS. To overcome this vexing issue and lift the exponential degeneracy, we added weak Gaussian perturbations of standard deviation of $\sigma_p = 0.05$ to the otherwise uniform unit (absolute value) strength nearest neighbor couplings.

In Fig. S5, we present the results of the LSBS on these perturbed fully-frustrated systems. The fully-frustrated square and honeycomb lattice spin systems exhibit characteristics similar to those of the spin glass, where the error rate approximately follows a power law with respect to L_{sub} . The right and left panels of Fig. S5 provide, respectively, a side by side comparison of the perturbed fully frustrated spin systems with their spin-glass counterparts. The found exponents κ in the power law form for the error rate of Eq. (2) are $\kappa_{\text{square}} = 0.685(8)$, $\kappa_{\text{square,FF}} = 0.73(1)$, $\kappa_{\text{honeycomb}} = 0.67(3)$, $\kappa_{\text{honeycomb,FF}} = 0.66(2)$ are nearly matching for the honeycomb and square lattices. For the lowest perturbation strength shown ($\sigma_p = 0.05$), the fully-frustrated triangular lattice spin system does not exhibit a power law decay of the average LSBS error in L_{sub} while its spin-glass counterpart obeys Eq. (2) with $\kappa_{\text{triangle}} = 0.55(1)$. In Fig. S6, we show the different performance of LSBS for the perturbed ($\sigma_p = 0.05, 0.20, 0.40$) fully frustrated triangular lattice spin system. As σ_p increases, the difficulty encountered by the LSBS decreases accordingly. We speculate that this may be because the original (unperturbed uniform) system's 'degeneracy' is further split as σ_p grows in size. For the strongest perturbation $\sigma_p = 0.40$ investigated, we found $\kappa_{\text{triangle,FF}} = 0.79(4)$ while

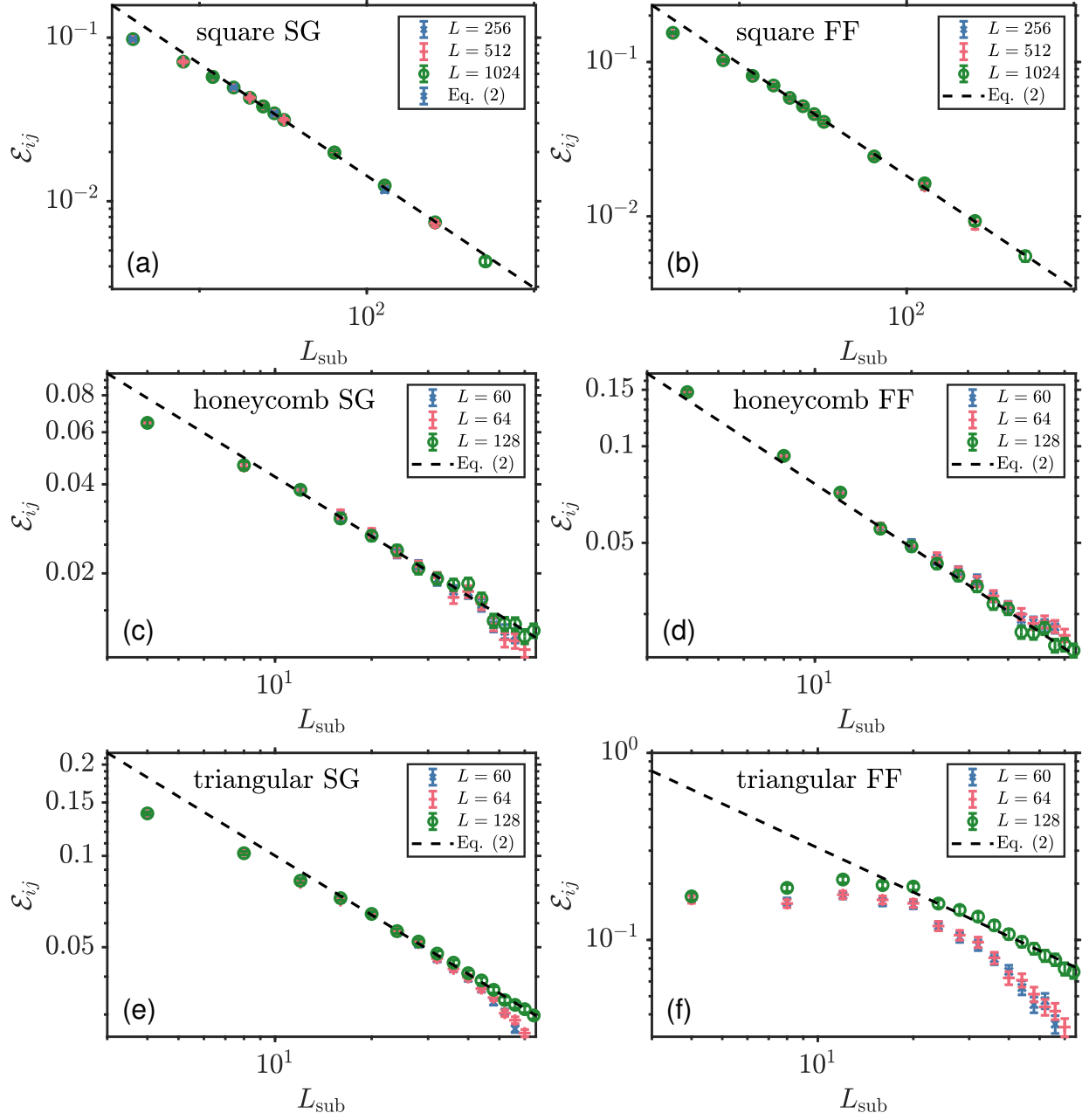


FIG. S5: A side by side comparison of the performance of LSBS on 2D spin-glass systems (left) and their weakly perturbed ($\sigma_p = 0.05$) fully-frustrated counterparts (right). Shown are the results for square lattice spin-glasses and the perturbed fully frustrated square lattice systems (panels (a) and (b) respectively), honeycomb (c,d), and triangular lattice (e,f) systems. For this weak perturbation σ_p , except for the fully-frustrated triangular lattice systems, the error rate decays as a power of L_{sub} . (Results for stronger perturbations of the fully-frustrated triangular lattice system are given in Fig. S6.) $\kappa_{\text{square}} = 0.685(8)$, $\kappa_{\text{square,FF}} = 0.73(1)$, $\kappa_{\text{honeycomb}} = 0.67(3)$, $\kappa_{\text{honeycomb,FF}} = 0.66(2)$, $\kappa_{\text{triangle}} = 0.648(9)$, $\kappa_{\text{triangle,FF}} = 0.79(4)$.

for the triangular lattice spin-glass $\kappa_{\text{triangle}} = 0.648(9)$ (Fig. S5). Note that all fittings were performed for $L_{\text{sub}} \geq 16$ to minimize the finite size effect. We note that for different lattices, a fixed value of L_{sub} corresponds to a different number of spins N_{sub} the subsystem solver examines. The $\chi^2/\text{d.o.f}$ values for the fittings in Fig. S5

(a)-(f) are, respectively, 0.821, 0.770, 0.960, 0.836, 0.558, and 1.282.

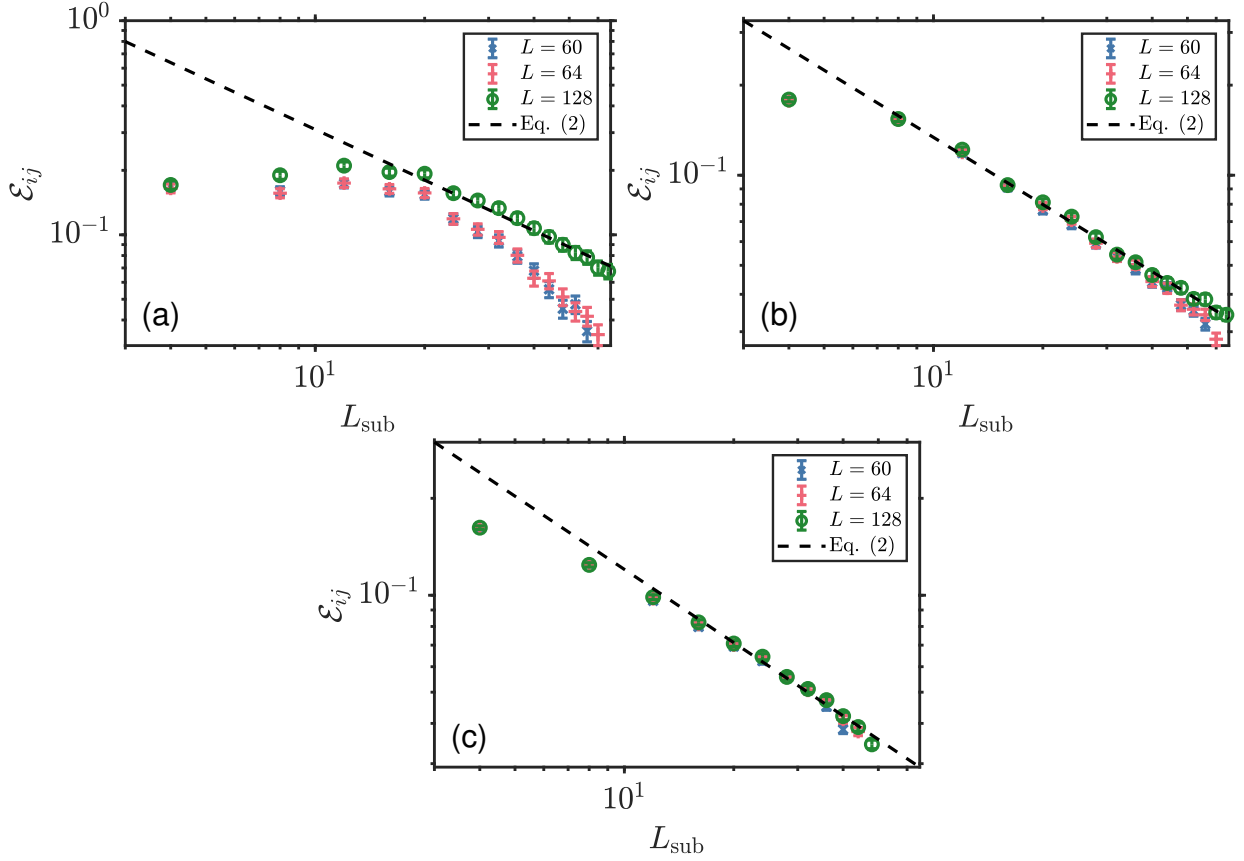


FIG. S6: Fully frustrated triangular lattice systems with increasing perturbation strength σ_p of their coupling constants. In panels (a,b,c), we respectively show the results for perturbations of strength $\sigma_p = 0.05, 0.20, 0.40$ (in units of the uniform absolute value of the unperturbed coupling constants of the fully frustrated system). The dashed lines correspond to the fit of Eq. (2) for triangular spin-glass systems (see bottom two panels of Fig. S5). The found exponents κ for these fittings are 0.79(4), 0.75(2), 0.76(3), with $\chi^2/\text{d.o.f} = 1.282, 0.590, 2.003$.

S7. SCALING OF THE SUBSYSTEM CRITICAL THRESHOLD

A. Boundary Conditions

In the main text, we considered how changing the value of one bond induces a change for the critical thresholds $\Delta J_{c,ij}$ of other bonds a distance r away. Here, we examine another question: If we change the boundary condition, instead of solely tuning one bond to its critical value, what will the change of the critical threshold of the central bond look like?

Towards this end, we studied $4 \leq L \leq 128$ square and $4 \leq L \leq 11$ cubic lattice systems. We first calculated the critical threshold $J_{c,ij}$ of the original system. Subsequently, we let all the spins σ_i on the boundaries to assume random values of ± 1 and then calculate the $J_{c,ij}$ of the central bond in this new system. The disorder averaged magnitude of change $[\Delta J_c]$ of the central bond is displayed in Fig. S7. Somewhat similar to the change of critical couplings when only a single bond is changed

that was discussed in the main text (Eq. (4)) and End Matter (Fig. 5)), we found a power-law decay

$$[\Delta J_{c,ij}] = (L/\ell_L)^{-a_L}, \quad (\text{S2})$$

with $\ell_{L,2D} = 1.73(2)$, $a_{L,2D} = 0.697(4)$ ($\chi^2/\text{d.o.f} = 0.544$) and $\ell_{L,3D} = 4.7(1)$, $a_{L,3D} = 0.36(2)$ ($\chi^2/\text{d.o.f} = 0.080$) for the square and cubic systems respectively (see Fig. S7.) The 2D exponent is close to that found (0.7 ± 0.02) for the dependence of the error rate of a patch solver on its size [8]).

B. Subsystem Size

In Fig. S8, we show the deviation of the critical threshold coupling of the central bond in the subsystem relative to the true (global) critical threshold value of this bond in the full system as a function of the subsystem size. Here, too, we find an algebraic decay,

$$[J_c - J_c^{\text{sub}}] \sim (\ell_J/L_{\text{sub}})^{\kappa_J}. \quad (\text{S3})$$

The rapid drop of this deviation with subsystem size further rationalizes our observed decreasing error rate with increasing subsystem size of Eq. 2 (see also cartoon of Fig. S2 explaining how such a small deviation favors accurate LSBS outcomes). Since the LSBS errors are incurred by a shift in the value of the subsystem J_c^{sub} relative to its true value of $J_{c,ij}$, it is natural to expect that the error rate $\mathcal{E} \propto |J_c - J_c^{\text{sub}}|$ for small $|J_c - J_c^{\text{sub}}|$. Indeed, within our numerical error, the exponents κ_J found via the fit of Eq. (S3) conform with those for the error rates (Eq. (2)) on different lattices,

$$\kappa_J = \kappa. \quad (\text{S4})$$

Specifically, for the 2D (square lattice) case, $\kappa_J = 0.684(7)$ (Fig. S2) while $\kappa = 0.685(8)$ (Table I). Similarly, in 3D (cubic lattice) systems, $\kappa_J = 0.17(2)$ while $\kappa = 0.18(3)$.

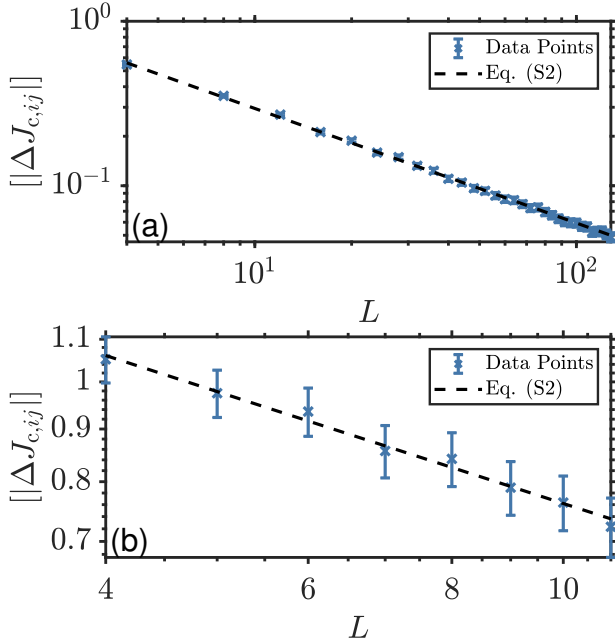


FIG. S7: After randomly resampling all spins on the boundaries, we calculated the average change of the critical threshold $[\Delta J_c]$ of the central bond as a function of system size L . This change is well-fitted by the power-law of Eq. (S2), where for (a) 2D systems- $\ell_{L,2D} = 1.73(2)$ and $a_{L,2D} = 0.697(4)$, and in (b) 3D systems- $\ell_{L,3D} = 4.7(1)$ and $a_{L,3D} = 0.36(2)$.

S8. SOME DETAILS OF THE NUMERICAL EXPERIMENTS

When we refer to simulations performed on a system with linear size L , it should be clarified that the system is not strictly square in shape. For instance, a two-dimensional system with $L = 1024$ actually corresponds

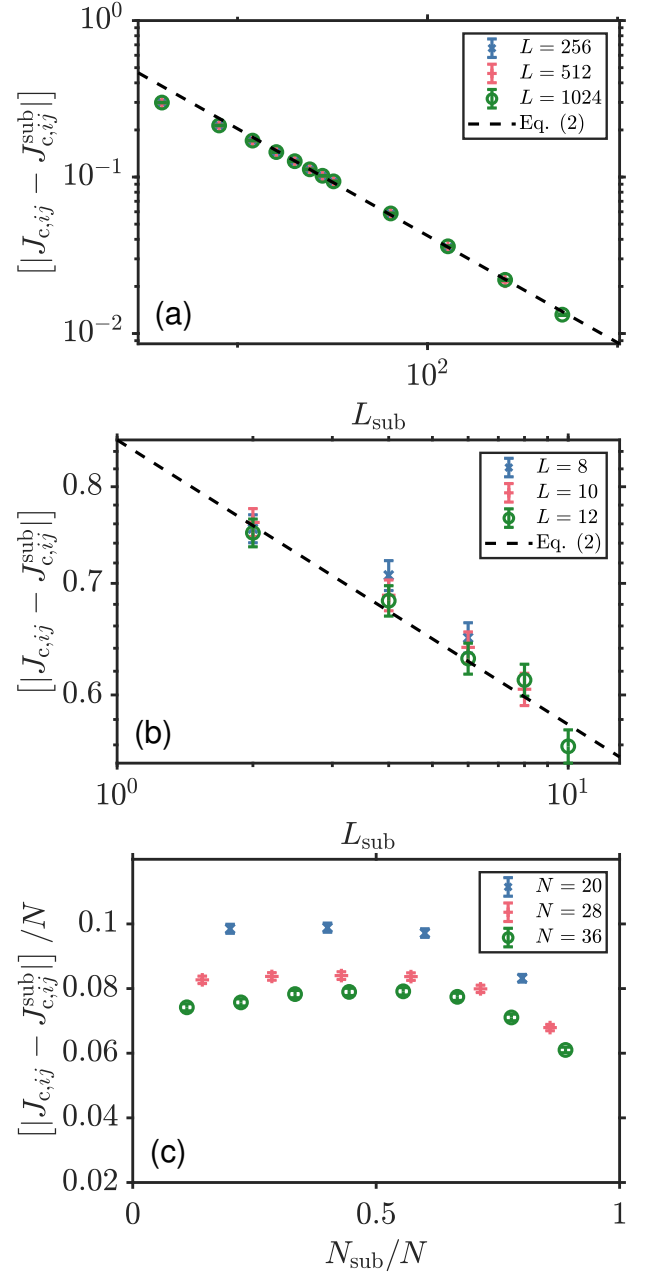


FIG. S8: For (a) the square and (b) cubic lattice EA model, the relation between $[J_c - J_c^{\text{sub}}]$ and L_{sub} also follows a power law and has a very similar fitting coefficient compared to $\mathcal{E}_{ij}$, Eq. (S3). In Eq. (S3), for 2D systems $\ell_J = 0.98(4)$ and $\kappa_J = 0.684(7)$; for 3D, $\ell_J = 0.4(1)$ and $\kappa_J = 0.17(2)$. The $\chi^2/\text{d.o.f}$ values for (a) and (b) are 11.698 and 1.823 respectively. (c) The SK model. For the 2D case, although we have adopted a cutoff $L_{\text{sub}} \geq 16$, we still obtained a relatively large $\chi^2/\text{d.o.f}$ value. When increasing the L_{sub} cutoff to 32 and 64, the corresponding $\chi^2/\text{d.o.f}$ value decreases to 4.276 and 2.650, with $\ell_J = 1.12(4), 1.20(5)$ and $\kappa_J = 0.703(6), 0.713(6)$.

to a lattice of 1025×1026 spins, just as in Ref. [1]. This tiny adjustment is made in order to ensure that the specific central bond is placed precisely at the center of the system.

Next, we discuss the strategy for subsystem sampling. A standard procedure, which we refer to as *parallel sampling*, proceeds as follows: one first randomly samples an entire system of size $L = 1024$, and then selects a subsystem of size $L_{\text{sub}} = 512$ for analysis. This process yields one data point. To obtain a data point at a smaller subsystem size, e.g., $L_{\text{sub}} = 256$, one would ideally resample the entire system and then extract the smaller subsystem. However, this approach can be computationally expensive, particularly when data across multiple L_{sub} values are needed.

To mitigate computational costs while still acquiring a sufficient number of data points for statistically reliable results (e.g., LSBS error rate), we adopt an alternative procedure, referred to as *sequential sampling*. In this approach, a single $L = 1024$ system is sampled, and multiple nested subsystems, such as $L_{\text{sub}} = 512, 256$, and 128 , are subsequently extracted and analyzed from the same entire system. Although the measurements obtained from these subsystems are theoretically correlated, this correlation should have minimal impact on our final results. For example, as shown in Fig. S9, we compare the fits to Eq. (2) obtained for $L = 1024$ and $L_{\text{sub}} = 512$, and the results are in close agreement—both visually and in terms of the fitted parameters: $\ell_{\mathcal{E},\text{parallel}} = 0.18(3)$, $\kappa_{\text{parallel}} = 0.67(2)$, $\ell_{\mathcal{E},\text{sequential}} = 0.20(1)$, $\kappa_{\text{sequential}} = 0.685(8)$.

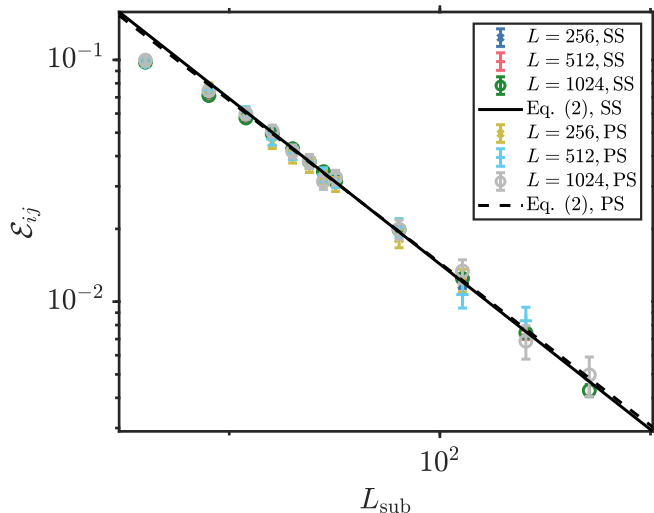


FIG. S9: Comparison of the fit to Eq. (2) for subsystem data extracted using the sequential and parallel sampling strategy respectively. Both curves correspond to systems with total size $L = 1024$ and subsystem size $L_{\text{sub}} = 512$. SS and PS are short for Sequential Sampling and Parallel Sampling respectively.

S9. EXACTLY SOLVABLE LIMITS

In what follows, we discuss two limits in which an exact analysis is possible: (1) 1D systems and (2) large n (or spherical model) realizations. These simply exactly solvable models will yield results notably different from those that we found for the 2D and 3D Ising EA spin-glasses.

A. 1D Ising spin-glass

One-dimensional spin-glass systems are rather special in many regards (including their exceptionally trivial exact solvability). Here, the critical thresholds exhibit distinct behaviors for periodic boundary conditions (PBC) and free (or open) boundary conditions (FBC). For the free boundary condition, all critical thresholds $J_{c,ij} = 0$, ensuring the locality of $J_{c,ij}$. For the periodic boundary condition, we define $J_{c,ij} \equiv \zeta |J_{c,ij}|$, with $\zeta = \pm 1$ indicating the sign of $J_{c,ij}$. The value of ζ is chosen such that the system is *frustrated*, i.e., contains an odd number of antiferromagnetic $J_{ij} < 0$ bonds. In the frustrated system, the GS is determined by having the smallest absolute bond being unsatisfied. Consequently, $|J_{c,ij}| = \min_{(i,j) \neq (i_0,j_0)} |J_{ij}|$. Obviously, both the sign and the magnitude of $J_{c,ij}$ are determined non-locally. In summary, we found that the locality holds for the FBC case, but not for the PBC case.

B. Spherical Model Spin-Glasses

Another analytically solvable limit is that of the spherical [9] (or large n) soft spin-glass model [10] counterpart of the Ising spin-glass Hamiltonian of Eq. (1). This model is defined by the following Hamiltonian and global constraint,

$$H = - \sum_{\langle ij \rangle} J_{ij} s_i s_j, \quad \sum_i s_i^2 = N. \quad (\text{S5})$$

Unlike the N Ising spins $\{\sigma_i\}_{i=1}^N$ of Eq. (1), the N spins $\{s_i\}_{i=1}^N$ are now *arbitrary real numbers* subjected only to the single *global normalization* constraint in Eq. (S5) concerning the sum of their squares. That is, only the mean (over the entire N spin system) value of s_i^2 is constrained to be unity unlike the Ising system where N constraints of the type $\sigma_i^2 = 1$ appear for each of the N spins σ_i . We find that this global constraint removes the otherwise inherent locality of the critical thresholds. The Hamiltonian of Eq. (S5) is a bilinear in the real variables s_i and thus the GS can be found by diagonalizing the real symmetric $N \times N$ matrix whose elements are the couplings J_{ij} . Following the below theorem, we may readily establish that for any bond $\langle ij \rangle$ in the system (keeping all other bonds fixed), the product $s_i s_j$ may vanish in

a single interval of J_{ij} values (including possibly only a single coupling).

Theorem. Given a real symmetric matrix $A_{m \times m}$, consider the perturbation $A \rightarrow A' \equiv A + \epsilon P$, with $P_{12} = P_{21} = +1$, $P_{ij} = 0$ otherwise, and $\epsilon > 0$. The normalized principle eigenvectors $v \equiv (v_1, v_2, \dots, v_m)^T$ of A and v' of A' (with respective eigenvalues λ and λ') satisfy $v_1 v_2 \leq v'_1 v'_2$.

Proof. Note that $v_1 v_2 = v^T P v / 2$, therefore we only need to prove $v^T P v \leq v'^T P v'$. By the definition of the vector v as the (normalized) principle eigenvector of A ,

$$v^T A v \geq v'^T A v' \quad (\text{S6})$$

$$v^T (A + \epsilon P) v \leq v'^T (A + \epsilon P) v' \quad (\text{S7})$$

Plugging Eq. (S6) into Eq. (S7), $v^T P v \leq v'^T P v'$. ■

This theorem formally establishes (the intuitively expected behavior) that the two-spin product $s_i s_j$ is monotonically non-decreasing in the coupling J_{ij} between them. (In the respective asymptotic $J_{ij} \rightarrow \pm\infty$ limits, the monotonic behavior saturates and the product $s_i s_j = \pm \frac{N}{2}$.) The demonstrated monotonicity ensures that we can define a *single interval* $[J_c^{(L)}, J_c^{(H)}]$ (that may possibly be a point of zero measure) for which the nearest-neighbor product $s_i s_j$ vanishes in the GS of Eq. (S5). Here, $J_c^{(L)} \equiv \sup\{J_{ij} | s_i s_j < 0\}$ and $J_c^{(H)} \equiv \inf\{J_{ij} | s_i s_j > 0\}$. In the following, we will use the shorthand $J_{c,ij}$ to denote the interval $[J_c^{(L)}, J_c^{(H)}]$. Unlike the Ising variant studied in the main paper text that had a single point value of the critical coupling, a general interval $[J_c^{(L)}, J_c^{(H)}]$ will not be of vanishing measure.

1. Symbiosis and Competition

Unlike the discrete Ising model of Eq. (1), within the spherical model, the influence of one bond on another bond's critical threshold $J_{c,ij}$ is rather trivial. As we illustrate in Fig. S10, even if two bonds share the same Euclidean distance to a modified bond, other facts are at play. To highlight this difference, the following extremal analysis would be very useful. In a 3D, $L = 10$ system, setting the blue bond to be $J_0 = +100.0$ as a 'strong' bond and we tune the value of the green bond and the red bond respectively, from $-2J_0$ to $2J_0$, we see quite different behaviors of $s_i s_j$, see the top side of Fig. S11. Now, we present two models to explain this behavior.

We neglect most of the sites and interactions and then we write down the model of "symbiosis" (green-blue) and "competition" (green-red): $H \simeq H_S = -J_{13}s_1 s_3 - J_{12}s_1 s_2$ and $H \simeq H_C = -J_{3'4}s_{3'} s_4 - J_{12}s_1 s_2$. Here we label the blue, green, and red bonds as $\{1, 2\}$, $\{1, 3\}$, $\{3', 4\}$, for the convenience of the matrix representations. Obviously, these bilinear forms appearing in the Hamiltonian

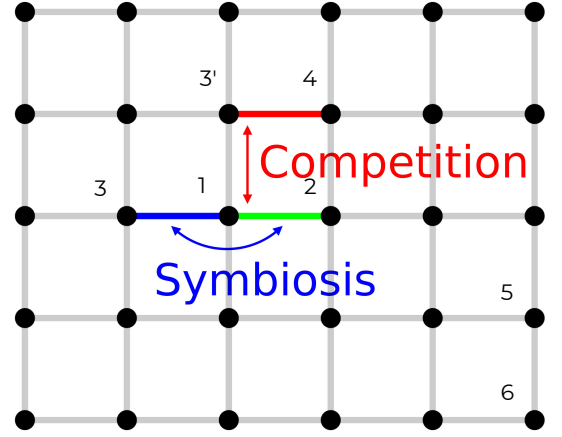


FIG. S10: A sketch of symbiotic (green-blue) and the competitive (green-red) bonds in the spherical model spin-glass, see text. We underscore that *irrespective of how far away they are*, all bonds (e.g., bond (5, 6)) that do not share a common spin with the central (green) bond exhibit an identical competitive relation with it.

can be expressed as $H_{S|C} = -\frac{1}{2} J_0 s^T (A_{S|C}) s$,

$$A_S = \begin{bmatrix} 0 & 1 & \alpha \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{bmatrix}, A_C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & 0 \end{bmatrix}. \quad (\text{S8})$$

Here, the subscript S|C denotes symbiosis or competition. In the context of the symbiosis captured by H_S , the vector $s^T = (s_1, s_2, s_3)$ while for the competition H_C , the vector $s^T = (s_1, s_2, s_{3'}, s_4)$. The constant α is the ratio of the J_{ij} values for blue/red bonds to the green bonds, J_{ij}/J_0 . For A_S , the principal eigenvector is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2\alpha^2+2}}, \frac{\alpha}{\sqrt{2\alpha^2+2}})^T$. For A_C , the principle eigenstate is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T$ for $-1 < \alpha < 1$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ for $\alpha = \pm 1$, $(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T$ for $\alpha < -1$, and $(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$ for $\alpha > 1$. These eigenstates can provide a fairly good explanation for the behavior of the spin product $s_1 s_3$ or $s_{3'} s_4$ (see the upper part of Fig. S11).

Note that for any bond that does not share a spin with the bond (1, 2) in Fig. S10, for example, the bond (5, 6), no matter how far away they are, it will exhibit an identical competitive relationship as the bond (3', 4), with the bond (1, 2). This can be verified by, for example, setting $J_{12} = +100.0$ and all other bonds $J_{ij} = +50.0$ and observing $s_i s_j$, see Fig. S12. It can also be verified by showing the distribution of $|\Delta J_c^{(H)}|$ as $J_{i_0 j_0}$ changes from 0 to +100, see Fig. S13.

Now, let us go back and consider the impact of the strong green bond on $J_{c,ij}$ of the other bonds. For the blue bond, according to the symbiosis model, there is a

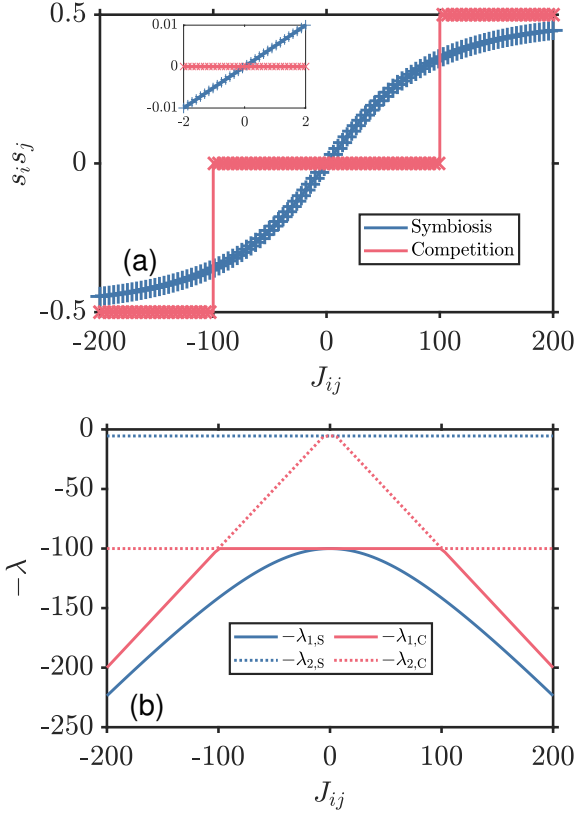


FIG. S11: (a) Computations for the spherical spin-glass model on an $L = 10$ cubic lattice. We calculate nearest-neighbor spin products, namely $s_1 s_3$ of the blue bond, and $s_3' s_4$ of the red bond, as a function of their respective couplings J_{ij} , while fixing the coupling of the green bond to be $J_0 = +100.0$. Blue and red scattering crosses represent the actual results from numerical simulation. Blue and red continuous curves show the predictions from the symbiosis and competition models given in Eq. (S8). The inset shows more data points within a smaller range of J_{ij} around zero. (b) On the same system, numerically we show the opposites of the first (solid) and second (dashed) largest eigenvalues $-\lambda$ of the system in symbiosis (blue, $-\lambda_{1(2),S}$) and competition (red, $-\lambda_{1(2),C}$) model respectively. Degeneracy is absent in the symbiosis model. In the competition model, a pair of degenerate points emerge when $|J_{ij}|$ of the red bond (Fig. S10) is similar to $|J_{ij}|$ of the blue bond. Just like the top side, the theoretical curve and the numerical curve (points) almost completely overlap, so to prevent clutter here, we only plotted the numerical curve.

nonzero slope at the origin for $s_1 s_3$. Therefore, when we consider the influence of other bonds as noise on it, these influences do not significantly move its zero point. On the other hand, for the red bonds, $s_3' s_4$ will remain approximately 0 over a considerable range. As a result, other bonds can easily disturb its zero point, causing $s_3' s_4$'s

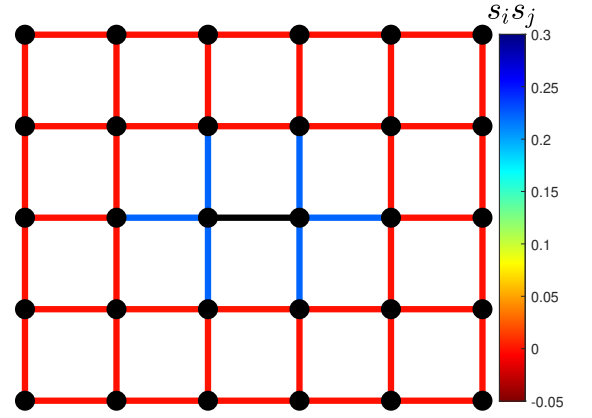


FIG. S12: A spherical spin glass mode having the central black bond coupling $J_{i_0 j_0} = +100$ with all other couplings J_{ij} being $+50$. The values of the computed GS spin products $s_i s_j$ are then color coded (see legend at right).

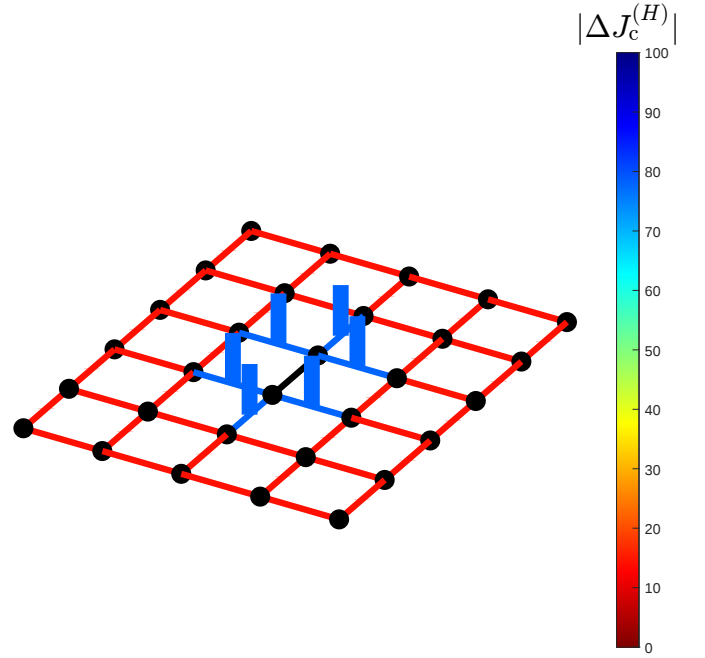


FIG. S13: Calculated change of the critical threshold ($|\Delta J_c^{(H)}|$) when tuning the central bond strength $J_{i_0 j_0}$ from 0 to 100. Each bond in a small 5×6 square lattice system with initial bond strengths is drawn from the symmetric Gaussian distribution (i.e., that centered about a mean of $\bar{J} = 0$). We find that for the continuous spins in the spherical model (unlike the discrete spins in the Ising spin-glass model in which there is a single critical coupling), there is, generally, a *range of critical threshold coupling constant values*. The displayed $|\Delta J_c^{(H)}|$ are associated with minima of the gap between the principle eigenvalue and the second largest eigenvalue (see text).

zero point to potentially move far away from zero. This implies that despite the same Euclidean distance, the impact of the blue bonds on the $J_{c,ij}$ of the green and red bonds differs considerably.

We can also consider an alternative definition for $J_{c,ij}$, which is to set $J_{ij} = J_{c,ij}$ (which could be a point, a pair of points, or an interval) in such a way that the difference between the principal eigenvalue and the second principal eigenvalue is minimized. It is worth noting that in the limiting case illustrated in the top and bottom sides of Fig. S11, these two definitions actually yield the same value(s) of $J_{c,ij}$. In fact, according to the models in Eq. (S8), we could easily get $-\lambda_{1,S} = -\sqrt{\alpha^2 + 1}$, $-\lambda_{2,S} = 0$; and $-\lambda_{1,C} = \min(\pm\alpha, -1)$, $-\lambda_{1,C} = \max(\pm\alpha, -1)$. Of course, to get similar curves on the bottom side of Fig. S11, one should replace α here with J_{ij}/J_0 and multiply the eigenvalues by J_0 .

What needs to be added is that this competition model is not limited to the red bond shown in Fig. S10. In fact, all bonds that are not directly connected to the green bond are in competition with the green bond. To be specific, similar behaviors can be seen in Fig. S11 for all these competitive bonds. Note that this conclusion holds for both definitions of $J_{c,ij}$. When the strength J_{ij} of the modified bond (green bond) increases in absolute value, see Fig. S14:

- For bonds having a symbiotic relationship, $J_{c,ij}$ will be more like a single point tending zero, that is, $J_c^{(L)} \rightarrow 0$, $J_c^{(H)} \rightarrow 0$.
- For bonds with a competition relationship, $J_{c,ij}$ will be more like an interval and asymptotically $J_c^{(L)} \rightarrow -J_0$, $J_c^{(H)} \rightarrow J_0$.

Overall, in the spherical model, the locality of the critical threshold does not exist like that in the Ising spin model. Specifically, although modifying the J_{ij} of a bond will usually affect other bonds' critical thresholds globally, the effects are not all the same. Symbiotic bonds and competitive bonds can be distinguished by whether they have the minimum topological distance to the modified bond, and they exhibit completely different behaviors regarding $J_{c,ij}$.

S10. SCALING ARGUMENT FOR OBSERVED EXPONENTS

Independent of any particular theory, as rigorously established in [11], the Gaussian EA spin-glass system exhibits a single GS pair that are related by a global spin inversion (so long as the continuum limit of continuous coupling is taken prior to the thermodynamic large system limit). This implies that in the $L \rightarrow \infty$ limit, the bond product $\sigma_i \sigma_j$ amongst nearest neighbor spins assumes a unique value (of either +1 or -1) in either of its GSs —

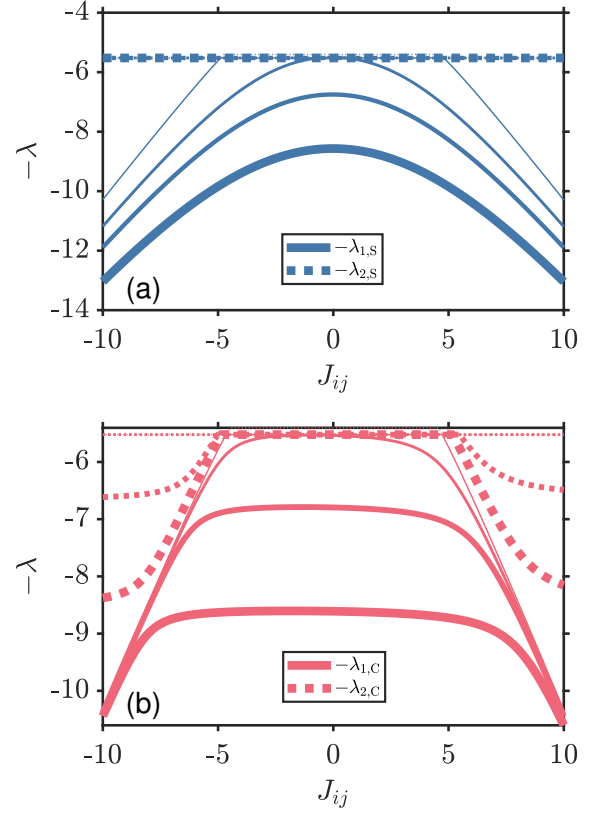


FIG. S14: (a)-(b) The sign inverted largest and second-largest eigenvalues $-\lambda_1, -\lambda_2$ when tuning the J_{ij} value of the symbiotic (blue) bond and the competitive (red) bond, on a $3D, L = 10$ system, with different J_{ij} values of the central (green) bond. Thicker curves correspond to stronger central bonds.

the correct value in the global GS pair. Thus, with $\langle \cdot \rangle$ denoting the GS average, the further disorder averaged $[\langle \sigma_i \sigma_j \rangle] = 1$. Our results indicate that the LSBS accuracy is independent of the system size L . This implies that we may consider the LSBS in the $L \rightarrow \infty$ limit, and examine the dependence of $[\langle \sigma_i \sigma_j \rangle]$ with, some abuse of notation, $\langle \cdot \rangle$ now denoting the subsystem GS average on L_{sub} with the external spin configurations averaged over. The variation of this disorder averaged GS spin product from unity arises solely due to the disorder average error rate $\bar{\mathcal{E}}_{ij}$ of a finite subsystem LSBS (now further averaged over external spin configurations that are consistent with the GS of the subsystem) that also samples incorrect (i.e., opposite sign values) of the bond spin product as compared to its value in the true original $L \rightarrow \infty$ rendition. That is,

$$[\langle \sigma_i \sigma_j \rangle] = 1 - \bar{\mathcal{E}}_{ij}. \quad (\text{S9})$$

Interestingly, for the square lattice in [8] the dependence of $[\langle \sigma_i \sigma_j \rangle]$ on a particular choice of the ratio between the system size and the subsystem size ($L/L_{\text{sub}} = 2$) was

examined when resampling the “shell” of the subsystem (thus emulating $\mathcal{E}_{ij}$). Our square lattice value of κ in Table I (0.685(8)) is close to the reported value (of 0.70 ± 0.02) in [8] for resampling subsystem shells. The two are possibly related by the two considerations listed below.

- (i) The proven uniqueness of the GS pair [11]. We underscore that this rigorous result concerning uniqueness of the GSs does not hinge on any particular theory or assumption. Rather, it always holds provided that the continuum limit of the spin couplings is taken prior to the thermodynamic limit.

and

- (ii) The independence of error rate $\mathcal{E}_{ij}$ on L that we establish in the current work (thus allowing us to consider the thermodynamic $L \rightarrow \infty$ limit).

In the current work, there are no external spins to the subsystem when the GSs are computed on the open subsystem while in [8] spin configurations external to the subsystem are resampled and averaged over. The calculation in the current work was a direct one on a small local subsystem with no regard to what the spin configurations in the external system might be. However, intuitively, the similarity between the two (that is, with $\bar{\mathcal{E}}_{ij}$ of Eq. (S9) possibly being equal to the error rate $\mathcal{E}_{ij}$ (Eq. (2)) that we focus on in the current work) is suggestive. The decay of the error rate with subsystem size indeed points to the robustness of the subsystem GS configurations to the inclusion of additional external spins. In what briefly follows we discuss what will transpire if we may consider the fraction of incorrect nearest neighbor bonds (i.e., the error rate $\mathcal{E}_{ij}$) as the fraction of bonds that lie on the boundary of a ZED of linear scale L_{sub} . The latter is the number of bonds flipped between two GSs and thus incorrect in the “original” GS relative to all nearest neighbor bonds in the subsystem. In such a situation then, extending the suggestions of [8], the value of κ in Eq. (2) may scale as

$$\kappa = d - d_s \quad (\text{S10})$$

with d_s the surface fractal dimension of the ZED. Reverting to the argument of [8] and trivially extending it to general dimensions, this is so since the number of “wrong” bonds in a region of linear size L_{sub} scales as $L_{\text{sub}}^{d_s}$ whereas the total number of bonds in that region $\sim L^d$ leading to a fraction of wrong bonds (the error rate) scaling as $L_{\text{sub}}^{d_s-d}$. Specifically, for our analyzed 2D (square lattice) systems $\kappa = 0.685(8)$ (Table I) that is numerically close to $(d - d_s) = 0.72$ given the fractal dimension of the ZED surface $d_s = 1.275(30)$ [1]. For the cubic lattice system (for which our numerical errors may be larger given the smaller system size that we are able to examine), the ZED fractal dimension $d_s = 2.76$ [1]; this suggests a value of $\kappa = 0.24$ as compared to our obtained value of $\kappa = 0.18$ (Table I) for our examined cubic lattice systems. The numerically observed Eq. (S4) suggests that the exponent κ_J in Eq. (S3) may, similarly, be

equal to $(d - d_s)$. Furthermore, given the rather universal character of the ZED volume and area distributions [1] and Eq. S10, near-universal (spatial dimension dependent yet specific lattice type independent) values of the error rate exponent κ may be anticipated for EA spin-glass systems. This is consistent with the values of κ that we found for different 2D lattices (square: 0.685(8), honeycomb: 0.67(3), and triangular (for which the largest deviation from our other investigated 2D lattice occurs): 0.648(9)).

S11. SPIN-GLASS CONSTRAINED DISORDERED-AVERAGE CRITICALITY

We conclude by discussing the prospect of disorder-averaged GS criticality at the critical threshold and thus general transitions (since any transition between GSs arises from varying couplings across their critical threshold). By “criticality” we allude here to algebraic deviations of gauge invariant (i.e., invariance of the Hamiltonian and associated distribution of coupling constants J_{ij} under the simultaneous transformations $\sigma_i \rightarrow \eta_i \sigma_i$, $J_{ij} \rightarrow \eta_i J_{ij} \eta_j$ with arbitrary local $\eta_i = \pm 1$) correlation functions from their asymptotic long distance limit.

In the following, $\langle \cdot \rangle$ denote averages over the set of all GSs for a fixed set of couplings when degeneracy arises—when one GS pair (i.e., two states related by the global inversion of all spins) of the system becomes degenerate with another pair. We wish to compute the two-point correlation function $G_{ij} \equiv \langle \sigma_i \sigma_j \rangle$ at the transition between the degenerate states (in systems having an unbiased probability distribution of J_{ij} , the GS pair averages $\langle \sigma_i \rangle = 0$). To consider gauge invariant quantities that do not vanish identically, we examine the disorder average $\Gamma_{i,j} \equiv [G_{ij}^2]$ [12]. Here, the disorder average is that over the set of all (gauge-invariant) couplings for which GS degeneracy arises. Now, within each element of the set of possible GS transitions across the critical threshold, with the said disorder average over all couplings where degeneracy arises following an internal average over all GSs at those couplings, we have, longhand,

$$\begin{aligned} \Gamma_{ij} &= [(\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle)^2] = [(\langle \sigma_i \sigma_j \rangle)^2] = \\ &= [(\chi_{i \in \text{ZED}} \chi_{j \in \text{ZED}} + \chi_{i \notin \text{ZED}} \chi_{j \notin \text{ZED}})^2] \\ &= [(\chi_{i \in \text{ZED}} \chi_{j \in \text{ZED}}) + (\chi_{i \notin \text{ZED}} \chi_{j \notin \text{ZED}})] \quad (\text{S11}) \end{aligned}$$

In Eq. (4), $\chi_{i \in \text{ZED}} = 1$ if i lies in the ZED and $\chi_{i \in \text{ZED}} = 0$ otherwise (with the opposite definition for $\chi_{i \notin \text{ZED}}$). In Eq. (S11) we employed the trivial observation that if both sites i and j lie in the ZED or both of these sites lie outside the ZED, then the product of the spins at these two sites ($\sigma_i \sigma_j$) assumes the same value (either “1” in all of these states or “−1” in all of the four degenerate states (two degenerate pairs of GSs); these uniform sign

values add coherently in the GS average. Thus, regardless of its sign, the average $\langle \sigma_i \sigma_j \rangle$ over all GSs (the latter four states) is of unit norm. The deviation of the GS and disorder averaged Γ_{ij} from unity is given by the probability that one of the sites i or j lies within the ZED with the other site (j or i , respectively) being outside the ZED. For a sequence of ZEDs connecting degenerate GSs (the general case), we consider the probability distribution associated with the “last” ZED.

We now consider the specific (constrained) case in which the degenerate GSs are such that $i \in \text{ZED}$ and turn to the asymptotic scaling of large distance $r = |i - j|$ of Γ_{ij} . Since the (disorder averaged) cumulative ZED volume distribution function is a power law [1], Γ_{ij} decays as a power law in the distance r . We earlier found [1] that the cumulative probability associated with ZED volume $|D|$ scaled as

$$P(|D| \geq \mathcal{V}) = 1 - F(\mathcal{V}) = \frac{1}{\mathcal{V}_0^{\kappa_v}} \Omega \left(\frac{\mathcal{V}}{\mathcal{V}_0} \right) \sim k_v \mathcal{V}^{-\kappa_v}. \quad (\text{S12})$$

Eq. (S12) implies that the probability density for the volume decays algebraically with an exponent $\kappa_v + 1$. Thus, denoting by $\tilde{P}_r$ the probability density for the ZED to be of linear size r , we have that $\tilde{P}_r dr \propto \mathcal{V}^{-(\kappa_v+1)} d\mathcal{V} \propto r^{-d(\kappa_v+1)} r^{d-1} dr = r^{-(d\kappa_v+1)} dr$. To find the probability of a ZED of size $\geq r$ that includes site j we integrate $\int_r^\infty dr' \tilde{P}_{r'}$. Thus [13],

$$\Gamma_r \propto r^{-d\kappa_v}. \quad (\text{S13})$$

Plugging in the exponents [1] κ_v , we find that for both the cubic and square lattices the correlations decay with an exponent $d\kappa_d \sim 0.4$.

Away from couplings at which degeneracy of different GS pairs arises, $\Gamma_{ij} = 1$ for all i and j . The same also holds true (sans the disorder average) for the GSs of the Ising ferromagnet.

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- [12] We may trivially, with no further change, implicitly average G_{ij} over sites i and j of fixed separation $|i - j| = r$ prior to taking the disorder average to obtain Γ_{ij} .
- [13] We underscore our underlying assumptions where this critical scaling holds: The system is at a degeneracy point brought about by fine tuning the couplings. The above calculation is valid when the site i is a finite fraction of the ZED linear size away from its center (for simplicity, we assume that i is at the origin for which the coupling is tuned [1]) and a disorder average is performed over all such coupling realizations for which degeneracy arises.

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