The Binomial Spin Glass

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To establish a unified framework for studying both discrete and continuous coupling distributions, we introduce the *binomial* spin glass. In this model, the couplings are the sum of m identically distributed Bernoulli random variables. In the continuum limit $m \to \infty$, this system reduces to a model with Gaussian couplings, while $m = 1$ corresponds to the $\pm J$ spin glass. We show that for short-range Ising models on d-dimensional hypercubic lattices the ground-state entropy density for N spins is bounded from above by $(\sqrt{d/2m} + 1/N) \ln 2$ confirming that for Gaussian couplings the degeneracy is not extensive. We further argue that the entropy density scales, for large m, as $1/\sqrt{m}$. Exact calculations of the defect energies reveal the presence of a crossover length scale $L^*(m)$ below which the binomial spin glass is indistinguishable from the Gaussian system. These results highlight the non-commutativity of the thermodynamic and continuous coupling limits.

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Spin glasses are extremely rich systems that have continued to surprise for many decades [1–4, 6–8, 12, 19–21]. They represent paradigmatic realizations of complexity that are abundant in nature and numerous combinatorial optimization problems [14]. Abstractions of spin-glass physics have been very fruitful in diverse fields. These range from the development of powerful computer science algorithms and the understanding of computational complexity classes [15–18] to concepts in protein folding [19] and neural networks [20]. Notwithstanding this success, several fundamental questions still linger. Some of the most basic open problems [21] concern the character of the low-lying states and, in particular, whether there are many incongruent [22] ground states in finite dimensional, short-range, spin glasses. It has long been known that spin-glass systems with discrete couplings may rigorously exhibit an extensive ground-state degeneracy [23, 24]. These results do not extend to continuous coupling distributions [25–29]. Nonetheless, the possible vanishing, in the thermodynamic limit, of spectral gaps points to the fact that one needs to distinguish localized from extended excitations in such systems, and only the latter can give rise to a multitude of states.

In this paper, we connect the $\pm J$ and the Gaussian spin glass models by introducing a new interpolating model, the *binomial* spin glass that has a tunable control parameter m. We establish bounds of the spectral degeneracy of the Ising system on bipartite graphs, which includes the usual Edwards-Anderson (EA) model with $\pm J$ ($m = 1$) and Gaussian ($m \to \infty$) couplings [1, 30]. We thus show that discrete (finite m) spin-glass samples exhibit an extensive ground-state degeneracy, while continuous ones ($m \to \infty$) become two-fold degenerate. The response of a square lattice of $N = L \times L$ sites to system-spanning perturbations induced by a change of boundary conditions suggests that the system effectively behaves like the continuous $m \to \infty$ model up to a crossover length scale $L^*(m)$ that diverges as $m \to \infty$.

We next introduce the *binomial Ising spin glass* Hamiltonian, defined on any graph of N sites [31], with a spin $s_{\mathsf{x}} = \pm 1$ at each site x,

$$H_m = -\sum_{\langle \mathsf{xy} \rangle} \mathcal{J}^m_{\mathsf{xy}} s_{\mathsf{x}} s_{\mathsf{y}} \equiv -\sum_{\alpha=1}^{\mathcal{L}} \mathcal{J}^m_\alpha z_\alpha. \tag{1}$$

Here, the sum is over sites x and y, defining a link $\alpha = \langle \mathsf{xy} \rangle$, and $\mathcal{L}$ denotes the total number of links. The *binomial coupling* for each link α, $\mathcal{J}^m_\alpha \equiv \frac{1}{\sqrt{m}} \sum_{k=1}^m J^{(k)}_\alpha$, is a sum of m copies (or "layers") of binary couplings $J^{(k)}_\alpha = \pm 1$, each with probability p of being $+1$ and $1-p$ of being -1. The probability distribution of $\mathcal{J}^m_\alpha$,

$$\tilde{P}(\mathcal{J}^m_\alpha) = \sum_{j=0}^{m} \binom{m}{j} p^{m-j} (1-p)^j \delta\left(\mathcal{J}^m_\alpha - \frac{m-2j}{\sqrt{m}}\right), \tag{2}$$

is a binomial. In the large-m limit, for general p, the distribution (2) approaches a Gaussian of mean $\sqrt{m}(2p-1)$ and variance $\sigma^2 = 4p(1-p)$. In particular, for $p = 1/2$, the distribution $\tilde{P}(\mathcal{J}^m_\alpha)$ approaches the standard normal distribution usually considered for the EA model [1].

To understand the degeneracies in the spectrum, we study the entropy density of the ℓ-th energy level,

$$\mathcal{S}_\ell \equiv \frac{\sum_{\{\mathcal{J}^m_\alpha\}} P(\{\mathcal{J}^m_\alpha\}) \ln D_\ell(\{\mathcal{J}^m_\alpha\})}{N}, \tag{3}$$

where D_ℓ is the degeneracy of the ℓ-th energy level ($\ell = 0$ [23] denotes the ground state while $\ell = 1$ is the first

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excited state, etc.). $P(\{\mathcal{J}_\alpha^m\}) = \prod_{\alpha=1}^{\mathcal{L}} \tilde{P}(\mathcal{J}_\alpha^m)$ is the probability of the coupling configuration.

In what follows we embark on a derivation of an upper bound on the ground state entropy density $\mathcal{S}_0$ and briefly discuss the entropy density of a general level $\mathcal{S}_\ell$. We restrict ourselves to bipartite graphs, where any closed loop encompasses an even number of links α. Consider two spin configurations $|\mathsf{s}\rangle \neq |\mathsf{s}'\rangle$ and evaluate their energy difference $\Delta E = E(\mathsf{s}) - E(\mathsf{s}')$. From Eq. (1),

$$\Delta E = -\sum_{\alpha=1}^{\mathcal{L}} \mathcal{J}_\alpha^m \Big(z_\alpha(\mathsf{s}) - z_\alpha(\mathsf{s}') \Big) = -2 \sum_{\alpha=1}^{\mathcal{L}} \mathcal{J}_\alpha^m n_\alpha, \quad (4)$$

with integers $n_\alpha = 0, \pm 1$ defined by the relation

$$n_\alpha \equiv \frac{z_\alpha(\mathsf{s}) - z_\alpha(\mathsf{s}')}{2}, \quad (5)$$

where $z_\alpha(\mathsf{s}) = s_\mathsf{x} s_\mathsf{y}$ is the product of spins on the link α. If $|\mathsf{s}\rangle$ and $|\mathsf{s}'\rangle$ are degenerate then, by definition, $\Delta E = 0$. A degeneracy only occurs for some realizations $\{\mathcal{J}_\alpha^m\}$ of the couplings, and Eq. (4) can be understood as a set of conditions for the couplings to ensure this.

Consider an arbitrary reference configuration $|\mathsf{s}\rangle$ of energy $E(\mathsf{s})$ and examine its viable degeneracy with the contending $2^N - 1$ other Ising spin configurations $|\mathsf{s}'\rangle$. Each of these configurations leads to a particular set of integers $\mathsf{C}_j = \{n_\alpha\}_j$, which form the set $\{\mathsf{C}_j\}_{j=1,2^N-1}^{|\mathsf{s}\rangle}$. A subset of those, $\mathsf{Sat}_{|\mathsf{s}\rangle} = \{\mathsf{C}_{j_1}, \mathsf{C}_{j_2}, \cdots, \mathsf{C}_{j_\mathcal{N}}\}$, will satisfy the degeneracy condition $\Delta E = 0$ in Eq. (4) for some coupling realizations. There are two types of solutions to the equation $\Delta E = 0$:

(i) $n_\alpha = 0, \forall \alpha$, or

(ii) $n_\alpha \neq 0$, for at least one link α.

It is straightforward to demonstrate that there is a single configuration $|\mathsf{s}'\rangle (\neq |\mathsf{s}\rangle)$ for which (i) $n_\alpha = 0, \forall \alpha$ [32]. This is the degenerate configuration $|\mathsf{s}'\rangle$ obtained by inverting all of the spins in the configuration $|\mathsf{s}\rangle$. To determine whether the degeneracy may be larger than two, we need to compute the probability $\mathcal{P}$ that constraints of type (ii) may be satisfied. While we cannot exactly calculate this probability for general N and m, bounds that we will derive suggest that $\lim_{N\to\infty} \lim_{m\to\infty} \mathcal{S}_\ell = 0$. As we will emphasize, different large m and N limits may yield incompatible results.

Constraints $\mathsf{C}_j \in \mathsf{Sat}_{|\mathsf{s}\rangle}$ are in *a one-to-one correspondence* with zero-energy interfaces [33], whose *size* is equal to the number g_j of non-zero integers in the set $\{n_\alpha\}_j$. That is, given a fixed reference configuration $|\mathsf{s}\rangle$ and a degenerate one $|\mathsf{s}'\rangle$, all nontrivial [i.e., type (ii)] solutions to Eq. (4) are associated with configurations where the product $s_\mathsf{x} s'_\mathsf{x}$ is equal to -1 in a non-empty set of sites $\mathsf{x} \in R$. To avoid the trivial redundancy due to global spin inversion, consider the states $|\mathsf{s}\rangle$ and $|\mathsf{s}'\rangle$ for which the spin at an arbitrarily chosen "origin" of the lattice assumes the value $+1$. These states are related via $|\mathsf{s}'\rangle = U_{\mathsf{s}'\mathsf{s}}|\mathsf{s}\rangle$, where the *domain-wall operator* $U_{\mathsf{s}'\mathsf{s}}$ is the product of Pauli matrices that flip the sign of the spins s'_x at the sites x where $|\mathsf{s}\rangle$ and $|\mathsf{s}'\rangle$ differ. Regions R are bounded by zero-energy domain walls that are interfaces dual to the links with $n_\alpha = \pm 1$, i.e., surrounding the areas R where the spins in $|\mathsf{s}\rangle$ an $|\mathsf{s}'\rangle$ have opposite orientation. Each satisfied constraint $\mathsf{C}_j \in \mathsf{Sat}_{|\mathsf{s}\rangle}$ is associated with a state $|\mathsf{s}'\rangle = U_{\mathsf{s}'\mathsf{s}}|\mathsf{s}\rangle$ that is degenerate with $|\mathsf{s}\rangle$ for some coupling realization(s).

We next formalize the counting of *independent domain walls* or clusters of free spins to arrive at an asymptotic bound on their number [Eq. (10)]. This will, in turn, provide a bound on the degeneracy. We define a *complete* set of *independent* constraints $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle} \subset \mathsf{Sat}_{|\mathsf{s}\rangle}$, of cardinality $\mathcal{M}$, to be composed of *all* constraints $\mathsf{C}_{\bar{\jmath}} \in \mathsf{Sat}_{|\mathsf{s}\rangle}$ that lead to *linearly independent equations* of the form of Eq. (4), $\Delta E = E(\mathsf{s}) - E(\mathsf{s}_{\bar{\jmath}}) = 0$, on the coupling constants $\{J_\alpha^m\}$ [33]. All constraints in $\mathsf{Sat}_{|\mathsf{s}\rangle}$ are a consequence of the linearly independent subset of constraints $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$. Each constraint $\mathsf{C}_{\bar{\jmath}} \in \mathsf{Sat}_{|\mathsf{s}\rangle}$ is associated with a domain wall operator $U_{\mathsf{s}_{\bar{\jmath}}\mathsf{s}}$ that generates a degenerate state $|\mathsf{s}_{\bar{\jmath}}\rangle = U_{\mathsf{s}_{\bar{\jmath}}\mathsf{s}}|\mathsf{s}\rangle$. If for a given coupling realization $\{\mathcal{J}_\alpha^m\}$ there are $M(\{\mathcal{J}_\alpha^m\}) \leq \mathcal{M}$ such independently satisfied constraints, then the states

$$|\bar{n}_1 \bar{n}_2 \cdots \bar{n}_M\rangle \equiv U_{\mathsf{s}_{\bar{1}}\mathsf{s}}^{\bar{n}_1} U_{\mathsf{s}_{\bar{2}}\mathsf{s}}^{\bar{n}_2} \cdots U_{\mathsf{s}_{\bar{M}}\mathsf{s}}^{\bar{n}_M} |\mathsf{s}\rangle, \quad (6)$$

($\bar{n}_i = 0, 1$) will include all of the spin configurations degenerate with $|\mathsf{s}\rangle$. Taking global spin inversion into account [i.e., type (i) solutions], the degeneracy of $|\mathsf{s}\rangle$ is

$$D_{\ell(|\mathsf{s}\rangle, \{\mathcal{J}_\alpha^m\})} \leq 2^{M(\{\mathcal{J}_\alpha^m\})+1}, \quad (7)$$

where, for a system defined by the coupling constants $\{\mathcal{J}_\alpha^m\}$, the index $\ell(|\mathsf{s}\rangle, \{\mathcal{J}_\alpha^m\}))$ denotes the level ℓ the state $|\mathsf{s}\rangle$ belongs to. The set $\{|\bar{n}_1 \bar{n}_2 \cdots \bar{n}_M\rangle\}$ may contain additional states not degenerate with $|\mathsf{s}\rangle$ [34].

After averaging over disorder, the expected number of the linearly independent satisfied constraints $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$ is

$$\langle M \rangle_m \equiv \sum_{\{\mathcal{J}_\alpha^m\}} \sum_{\mathsf{C}_{\bar{\jmath}} \in \overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}} P(\{\mathcal{J}_\alpha^m\}) \delta^{\{\mathcal{J}_\alpha^m\}}(\mathsf{C}_{\bar{\jmath}}) \equiv \sum_{\mathsf{C}_{\bar{\jmath}} \in \overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}} \mathcal{P}(\mathsf{C}_{\bar{\jmath}}). \quad (8)$$

Here, $\mathcal{P}(\mathsf{C}_{\bar{\jmath}})$ is the probability that a linearly independent constraint $\mathsf{C}_{\bar{\jmath}}$ is satisfied. The Kronecker $\delta^{\{\mathcal{J}_\alpha^m\}}(\mathsf{C}_{\bar{\jmath}})$ equals 1 if $\mathsf{C}_{\bar{\jmath}}$ is satisfied for the couplings $\{\mathcal{J}_\alpha^m\}$ and is zero otherwise. Let us bound the probability $\mathcal{P}(\mathsf{C}_{\bar{\jmath}})$ by taking the form (2) of the coupling distribution into account. From the definition of the couplings $\{\mathcal{J}_\alpha^m\}$, the sum in Eq. (4) can effectively be read as including a sum over layers $k = 1, \ldots, m$, which hence includes $g_{\bar{\jmath}} m$ nonzero terms. For general $m \geq 1$, and even $g_{\bar{\jmath}} m$, the probability that half of the nonzero integers $n_\alpha J_\alpha^{(k)}$ in Eq. (4) are $+1$ and the remainder are -1 is

$$\mathcal{P}(\mathsf{C}_{\bar{\jmath}}) = \binom{g_{\bar{\jmath}} m}{\frac{g_{\bar{\jmath}} m}{2}} \frac{1}{2^{g_{\bar{\jmath}} m}} < \frac{1}{\sqrt{g_{\bar{\jmath}} m}}. \quad (9)$$

(An odd number of integers that are each ± 1 cannot sum to zero, i.e., Eq. (4) cannot be satisfied for odd $g_{\bar{\jmath}} m$.)

From asymptotic analysis [35] and Eq. (9), the probability $\mathcal{P}(\mathsf{C}_{\bar{\jmath}})$ scales (for large m) as (and, for any m, is bounded by) $1/\sqrt{g_{\bar{\jmath}} m}$. Denoting by $g_{\min}$ the smallest possible value of $g_{\bar{\jmath}}$ for the graph/lattice at hand,

$$\langle M \rangle_m \leq \frac{\mathcal{M}}{\sqrt{g_{\min} m}}. \tag{10}$$

On a general graph, the number $\mathcal{M}$ of linearly *independent* constraints $\mathsf{C}_{\bar{\jmath}}$ on the coupling constants $\{\mathcal{J}_\alpha^m\}$ cannot be larger than their total number, $\mathcal{M} \leq \mathcal{L}$, i.e., the number of links $\mathcal{L}$ on this graph. Putting all of the pieces together, Eqs. (7) and (10) imply

$$\sum_{\{\mathcal{J}_\alpha^m\}} P(\{\mathcal{J}_\alpha^m\}) \ln D_{\ell(|\mathsf{s}\rangle,\{\mathcal{J}_\alpha^m\})} \leq (1 + \frac{\mathcal{L}}{\sqrt{g_{\min} m}}) \ln 2. \tag{11}$$

Trying to evaluate the l.h.s. of Eq. (11) we must take into account that whatever $|\mathsf{s}\rangle$ we pick might be a ground state for some coupling configurations, but will be an excited state for others. Hence we cannot directly infer a bound to the average ground-state entropy (or, in fact, the entropy of any other energy level) from (11). Since the inverse temperature $1/(k_B T) = \partial \ln D/\partial E$, however, the system's ground-state degeneracy for couplings $\{\mathcal{J}_\alpha^m\}$ is typically lower than (or equal to) that of any other level ℓ [36], i.e., $D_0 \leq D_\ell$. This monotonicity of $D(E)$ implies that, typically, $\mathcal{S}_0 N = \sum_{\{\mathcal{J}_\alpha^m\}} P(\{\mathcal{J}_\alpha^m\}) \ln D_0(\{\mathcal{J}_\alpha^m\}) \leq \sum_{\{\mathcal{J}_\alpha^m\}} P(\{\mathcal{J}_\alpha^m\}) \ln D_{\ell(|\mathsf{s}\rangle,\{\mathcal{J}_\alpha^m\})}$. Then, Eq. (11) yields

$$\mathcal{S}_0 \leq (\frac{\mathcal{L}}{N\sqrt{g_{\min} m}} + \frac{1}{N}) \ln 2. \tag{12}$$

The entropy density of a $p \neq 1/2$ system is lower than that of $p = 1/2$. Thus, Eq. (12) constitutes a generous upper bound on $\mathcal{S}_0$ for general p. We next consider the average of Eq. (11) over all possible 2^N reference spin configurations $|\mathsf{s}\rangle$ (i.e., the average of Eq. (11) over all of the individual states in each of the energy levels ℓ associated with the couplings $\{\mathcal{J}_\alpha^m\}$). Performing this average and invoking the monotonicity of $D(E)$ (i.e., the increasing number of states lying in progressively higher energy levels) suggests that the entropy density $\mathcal{S}_\ell$ of Eq. (3) of low-lying excited levels $\ell > 0$ is, typically, also bounded by the righthand side of Eq. (12). For d-dimensional hypercubic lattices with periodic boundary conditions, the ratio $\mathcal{L}/N = d$ while $g_{\min} = 2d$. Thus, $\mathcal{S}_0 \leq (\sqrt{d/2m} + 1/N) \ln 2$. Eq. (12) further suggests that, in the thermodynamic ($N \to \infty$) limit [37],

$$\mathcal{S}_0(m') \sim \sqrt{\frac{m}{m'}}\, \mathcal{S}_0(m) \quad \text{for finite} \quad m, m' \gg 1. \tag{13}$$

We now study the exact m dependence of the ground state entropies of the binomial model on the square lattice with periodic boundary conditions. To this end, we employed an implementation of the Pfaffian technique of counting dimer coverings of the lattice as discussed in Ref. [38], which is a generalization of earlier methods [39, 40] to the case of lattices of arbitrary genus and, in particular, the case of fully periodic lattices. In Fig. 1, we present the results for the ground-state entropy, averaged over 1000 coupling realizations for each lattice size. The data are well described by

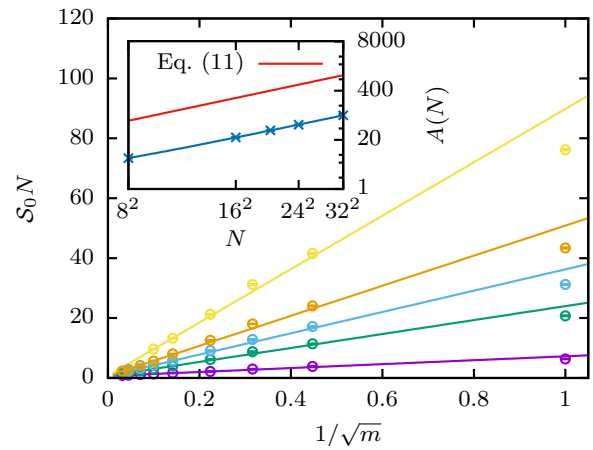

FIG. 1. Ground-state entropy $\mathcal{S}_0 N$ of the binomial Ising spin glass with m layers, cf. Eq. (1), on ($d = 2$) square lattices of $N = L^2$ spins from exact ground-state calculations (from the bottom: $L = 8$, 16, 20, 24, and 32). Lines are fits of the form of (14) to the data for sufficiently large m. The inset shows the linear scaling of the amplitude $A(N)$ of (15). The top line indicates the constraint imposed by the upper bound (12).

$$\mathcal{S}_0 N = \left(\frac{A(N)}{\sqrt{m}} + 1\right) \ln 2. \tag{14}$$

Linear fits in $1/\sqrt{m}$ for fixed N work well for sufficiently large m, as is illustrated by the straight lines in Fig. 1. Thus, for any finite N, as $m \to \infty$ the ground-state entropy is equal to $\ln 2$, implying a single degenerate ground-state pair. The slope $A(N)$ shown in the inset follows a linear behavior,

$$A(N) = aN + b, \tag{15}$$

and we find $a = 0.0858(4)$ and $b = 1.09(12)$. For not too small m, our data are hence fully consistent with

$$\mathcal{S}_0 = \left(\frac{a}{\sqrt{m}} + \frac{1}{N} + \frac{b}{N\sqrt{m}}\right) \ln 2. \tag{16}$$

When $N \gg \sqrt{m} \gg 1$, Eq. (16) is consistent with the physically inspired [37] scaling of Eq. (13). For large N, the bound of Eq. (12) would have been asymptotically saturated if $a \simeq 1$, far larger than the actual value of a. The behavior in the double limit $m, N \to \infty$ is subtle: while there is a unique ground-state pair if $m \to \infty$ before $N \to \infty$, several ground-state pairs are expected if $m, N \to \infty$ simultaneously while keeping $N/\sqrt{m}$ fixed.

Let us turn to the study of excitations. By construction, cf. Eq. (4), for finite m the energy is "quantized" in multiples of $1/\sqrt{m}$. It is therefore natural to expect a closing of the spectral gap as $m \to \infty$. That this is indeed

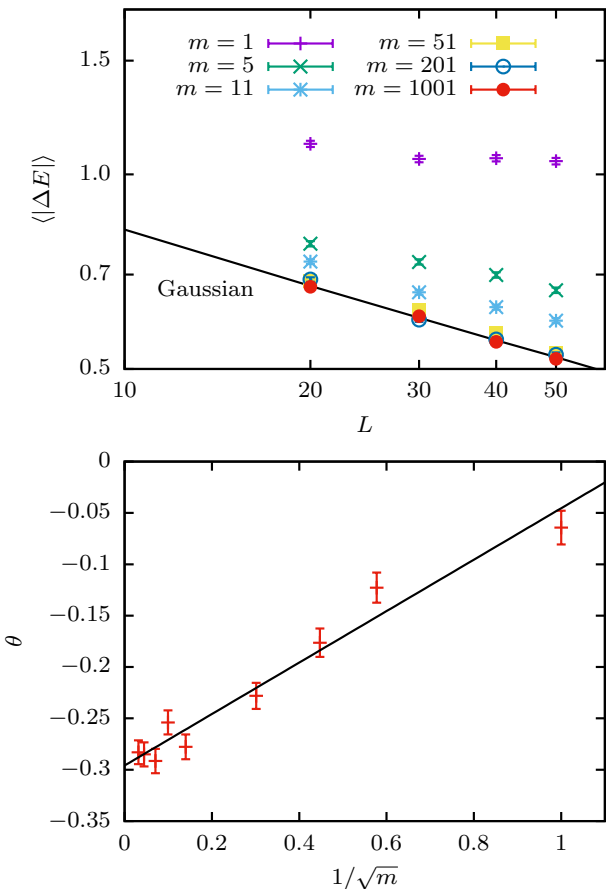

FIG. 2. Top: Defect energies for the binomial model with the indicated number of layers m, averaged over 10 000 disorder samples. The solid line shows the result for a Gaussian coupling distribution for comparison. Bottom: Effective spin stiffness exponents $\theta = \theta(m)$ resulting from fits of power law $\langle|\Delta E|\rangle = AL^\theta$ to the data shown in the top panel.

the case can be shown rigorously for the one-dimensional binomial spin glass in its thermodynamic limit, with different behaviors for odd and even m, see the discussion in the Supplemental Material [41]. The closing of the gap is a consequence of the existence of (rare) local excitations, i.e., finite-size clusters of almost free spins [42]. Whether gapless *non-local* excitations exist and which form they take in the thermodynamic limit is a long-standing question [43]. One possible approach of investigating such excitations consists of subjecting individual samples to a system spanning perturbation in the form of a change of boundary condition and studying how this affects the energy and configuration of the ground state. Such defect energy calculations [44] enable us to extract a scaling $\langle|\Delta E|\rangle \sim L^\theta$ of the defect energies with the spin stiffness exponent θ. Generalizing Peierls' argument [45–48] for the stability of the ordered phase, one should find $\theta > 0$ for cases where there is a finite-temperature spin-glass phase, and $\theta \leq 0$ otherwise. The latter case is expected for dimensions $d = 1$ and $d = 2$, whereas θ is positive for $d \geq 3$ [49, 50]. We employed techniques based on minimum-weight perfect matching [51, 52] to perform such calculations for the binomial model on the square lattice. The resulting disorder-averaged defect energies from *exact* ground-state calculations for samples with periodic and antiperiodic boundaries are shown in the top panel of Fig. 2. As m increases, the decay of defect energies as a function of L becomes steeper and the data approach the behavior of the Gaussian EA model. The effective spin stiffness exponents θ extracted from fits of the type $\langle|\Delta E|\rangle = AL^\theta$ are shown in the lower panel of Fig. 2. These exponents appear to interpolate smoothly between the limiting cases of the Gaussian model with $\theta = -0.2793(3)$ and the $\pm J$ system with $\theta = 0$ [49, 52]. Asymptotically, however, we expect that $\theta(m) = 0$ for any finite value of m when $N \to \infty$. This suggests a crossover length scale $L^*(m)$ below which the defect energies scale similarly to those in a continuous system while the discreteness of the spectrum becomes apparent only for system sizes beyond $L^*(m)$. In line with our earlier results for the degeneracy, the m dependence of the effective exponent θ suggests that different ways of taking the $m \to \infty$ and $N \to \infty$ limits yield *contrasting results* for the excitation energies. That is, if $N \gg (L^*(m))^2$ as $m, N \to \infty$, then the discrete nature of the model may manifest itself. Conversely, if $N \ll (L^*(m))^2$ when the double limit of $m \to \infty$ and $N \to \infty$ is taken, then the system may emulate the continuous EA model.

It is clear that if $\theta < 0$, as is the case for the Gaussian spin glass in two dimensions, excitations of a divergent length scale may entail a vanishing energy penalty. For $\theta \geq 0$, it is not enough to consider the average $\langle|\Delta E|\rangle$ and one should instead inspect the full probability distribution of domain wall energies and the weight it carries in the limit $\Delta E \to 0$ [42]. In how far such excitations correspond to incongruent states, however, one might only be able to infer by inspecting the configurations themselves.

In summary, we introduced and discussed the *binomial spin glass*. This model affords controlled access to the enigmatic continuous ($m \to \infty$) finite dimensional EA model. Its $m = 1$ realization is the quintessential discrete spin glass, the $\pm J$ model, for which many results are known. We derived bounds on the spectral degeneracy of the binomial spin glass on general lattices and graphs and suggested an asymptotic scaling that is fully supported by exact two-dimensional calculations. The behavior of the defect energies suggests the existence of a crossover length $L^*(m)$ below which the binomial model behaves like the Gaussian system. Our results show that various properties may depend on the order in which the thermodynamic ($N \to \infty$) and continuous coupling ($m \to \infty$) limits are taken. We remark that, for hypercubic lattices in d dimensions, the bound of Eq. (12) implies a two-fold degenerate ground state when $N\sqrt{d/m} \to 0$ (including the *thermodynamic limit* of the bonafide continuous EA spin glass in finite d). An exact analytical solution discussed in the supplemental material illustrates how the one-dimensional nearest neighbor spin glass becomes

gapless as $m \to \infty$ (with a different scaling of the gap for even and odd m).

We hope that our model will prompt further studies of ground-state degeneracies and excitations of spin-glass systems. These properties and their connection to the structure of the free-energy landscape have also recently attracted attention from another side. In the context of quantum annealing [53] as implemented in the devices by D-Wave and similar machines that are being developed by competing consortia, degeneracies are not a desired feature as the quantum annealing process does not sample such states uniformly [54]. On the other hand, continuous coupling distributions may also be undesired because of increased susceptibility to external noise implied by chaos in spin glasses [55–58]. Our binomial glasses may allow for realizations that suffer the least from these combined problems. While we focused on Ising models, the binomial spin glass can be defined for arbitrary spins, even quantum ones. For instance, the $O(n)$ *binomial spin glass* is defined by Eq. (1) with the product $s_{\mathsf{x}} s_{\mathsf{y}}$ replaced by a scalar product $\vec{s}_{\mathsf{x}} \cdot \vec{s}_{\mathsf{y}}$ between n component vectors of unit norm.

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Supplemental Material for The Binomial Spin Glass

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A. General Background and Motivation

The quintessential short-range Ising spin glass system is the Edwards-Anderson (EA) model, where at each lattice site x lies a classical spin $s_{\mathsf{x}} = \pm 1$, that interacts with nearest-neighbor spins only [1]. In the *discrete* binary version, the random couplings may assume only the two values $\pm J$. Conversely, the couplings are *continuous* random Gaussian variables in the *continuous* EA model. While the extensive ground state degeneracy is well established for various binary distributions, the situation for the continuous EA model has been mired by controversy. Parisi's tour de force solution [2] led to insights concerning the extensive nature of the ground state entropy of the infinite-range Sherrington-Kirkpatrick (SK) model [3]. The latter harbors a plethora of distinct thermodynamic states [4–6]. A measure of similarity between disparate thermodynamic states is provided by the well-known "overlap function" [4, 7, 8] $q_{rr'} = \frac{1}{N}\sum_{\mathsf{x}} \langle s_{\mathsf{x}}\rangle_r \langle s_{\mathsf{x}}\rangle_{r'}$, where N is the total number of lattice sites, and its average over the probabilities W_r and $W_{r'}$ of the realizations of the different pairs of states r and r' (the "overlap distribution function"), $P(q) = \sum_{rr'} q_{rr'} W_r W_{r'}$. The SK model displays a cascade of different overlaps (an ultrametric structure [9]) and replica symmetry breaking wherein $P(q)$ becomes nontrivial [10]. Standard ordered systems typically display a small number of symmetry related thermodynamic states (and zero temperature ground states) associated with a distribution $P(q)$ that is a sum of simple delta functions. While the Parisi solution and various related (effective infinite dimension or infinite range) mean-field treatments raise the possibility of an exponentially large number of ground states, other considerations [4, 11–18] suggest that in typical short-range spin glasses, there are (similar to ferromagnets) only two symmetry related ground states. Understanding of this question is not merely of academic importance; the behavior of real finite dimensional magnetic spin glass systems has long been of direct experimental pertinence, e.g., [19, 20].

In what follows, we explicitly define the standard EA model. Consider a general bipartite lattice (in any finite number of dimensions d) of size N, endowed with periodic boundaries, with an Ising spin s_{x} at each lattice site x. The EA spin glass Hamiltonian is given by

$$H = -\sum_{\langle \mathsf{xy}\rangle} J_{\mathsf{xy}} s_{\mathsf{x}} s_{\mathsf{y}} \equiv -\sum_{\alpha=1}^{\mathcal{L}} J_\alpha z_\alpha. \tag{S1}$$

The summation in Eq. (S1) is over nearest-neighbor spins at sites x and y sharing the link $\alpha = \langle \mathsf{xy}\rangle$, $z_\alpha = \pm 1$, and the total number of these links is $\mathcal{L} = d \times N$. In various standard Ising spin glass models, the spin couplings $\{J_\alpha\}$ in Eq. (S1) are customarily drawn from one of several well studied distributions. For instance, in the "binary Ising spin glass model" [21], the couplings $\{J_\alpha\}$ are random variables that assume the two values ± 1 with probabilities $P(J_\alpha = 1) = p$, $P(J_\alpha = -1) = 1 - p$ (i.e., a Bernoulli distribution). In the continuous EA model the couplings $\{J_\alpha\}$ are drawn from a Gaussian distribution of vanishing mean and variance equals to unity.

B. The trivial ground state pair given an assignment of link variables

Given the definition of the link variable $z_\alpha \equiv s_{\mathsf{x}} s_{\mathsf{y}}$, a moment's reflection reveals that

$$s_{\mathsf{y}} = s_{\mathsf{x}} \prod_{\alpha \in \Gamma_{\mathsf{xy}}} z_\alpha, \tag{S2}$$

where Γ_{xy} is any path on the lattice, composed of nearest-neighbor links, joining site x to site y. Thus, with $s_{\mathsf{y}}|_{|\mathsf{s}\rangle}$ denoting the value of the spin at site y in configuration $|\mathsf{s}\rangle$, we have that

$$s_{\mathsf{y}}|_{|\mathsf{s}\rangle} = s_{\mathsf{x}}|_{|\mathsf{s}\rangle} \prod_{\alpha \in \Gamma_{\mathsf{xy}}} z_\alpha|_{|\mathsf{s}\rangle}\ , s_{\mathsf{y}}|_{|\mathsf{s}'\rangle} = s_{\mathsf{x}}|_{|\mathsf{s}'\rangle} \prod_{\alpha \in \Gamma_{\mathsf{xy}}} z_\alpha|_{|\mathsf{s}'\rangle}. \tag{S3}$$

Now, if for all links α, the values of z_α are the same in both configurations $|\mathsf{s}\rangle$ and $|\mathsf{s}'\rangle$ (i.e., if $\{z_\alpha\}|_{|\mathsf{s}\rangle} = \{z_\alpha\}|_{|\mathsf{s}'\rangle}$) then, trivially,

$$\prod_{\alpha \in \Gamma_{\mathsf{xy}}} z_\alpha|_{|\mathsf{s}\rangle} = \prod_{\alpha \in \Gamma_{\mathsf{xy}}} z_\alpha|_{|\mathsf{s}'\rangle}. \tag{S4}$$

Taken together, Eqs. (S3) and (S4) imply that if, at a particular site x, the spin configurations $|\mathsf{s}\rangle$ and $|\mathsf{s}'\rangle$ share the same value of the spin, $s_{\mathsf{x}}|_{|\mathsf{s}'\rangle} = s_{\mathsf{x}}|_{|\mathsf{s}\rangle}$, then the spins must be identical at all other lattices sites y, $s_{\mathsf{y}}|_{|\mathsf{s}'\rangle} = s_{\mathsf{y}}|_{|\mathsf{s}\rangle}$. This, however, leads to a contradiction as $|\mathsf{s}'\rangle \neq |\mathsf{s}\rangle$. Therefore, if two distinct spin configurations satisfy condition (**i**) it must be that the respective spin values at any lattice site x are different, $s_{\mathsf{x}}|_{|\mathsf{s}'\rangle} = -s_{\mathsf{x}}|_{|\mathsf{s}\rangle}$. That is,

$$s_{\mathsf{y}}|_{|\mathsf{s}'\rangle} = -s_{\mathsf{y}}|_{|\mathsf{s}\rangle}, \quad \forall \mathsf{y}. \tag{S5}$$

Hence, if $n_\alpha = 0, \forall \alpha$ in Eq. (4), then there are, trivially, only two degenerate configurations ($|\mathsf{s}'\rangle \neq |\mathsf{s}\rangle$) related by a global spin inversion. The above simple proof

applies for arbitrary energy levels. Replicating, *mutatis mutandis*, the above argument to a general set of (non-necessarily vanishing) integers $\{n_\alpha\}$ over all lattice links α, illustrates that any set $\{n_\alpha\}$ may correspond to exactly two unique spin configurations.

C. Graphical Representation of the Constraints

In the main text we defined $\mathsf{Sat}_{|\mathsf{s}\rangle}$ to be the set composed of *all* constraints C_j satisfying the relation $\Delta E = E(\mathsf{s}) - E(\mathsf{s}') = 0$, in Eq. (4). We also defined the subset $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle} \subset \mathsf{Sat}_{|\mathsf{s}\rangle}$, comprising *all linearly independent* constraints. Here, we further introduce a restricted subset of constraints, that of *geometrically disjoint* and independent zero energy domain walls, $\overline{\mathsf{Sat}^{\mathsf{g}}}_{|\mathsf{s}\rangle} \subset \overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$. The subset $\overline{\mathsf{Sat}^{\mathsf{g}}}_{|\mathsf{s}\rangle}$ is defined by having no pair of different constraints on the coupling constants that involve links associated with the same lattice sites x.

In what follows, we provide a few simple examples illuminating the above definitions. To this end, we consider a 5×5 square lattice with binomial couplings $\{\mathcal{J}^m_\alpha\}$ (Fig. S1). We start with a random spin configuration $|\mathsf{s}\rangle$ (panel (a)). Panels (b) through (e), represent spin configurations $|\mathsf{s}'\rangle$ for which one or more spins are being flipped with respect to panel (a). The energy difference in each case can be easily calculated. For example,

$$\begin{aligned}\Delta E_{a,b} = E_a - E_b = -2(&\mathcal{J}^m_{19,14} n_{19,14} \\ &+ \mathcal{J}^m_{19,18} n_{19,18} + \mathcal{J}^m_{19,20} n_{19,20} + \mathcal{J}^m_{19,24} n_{19,24}),\end{aligned} \tag{S6}$$

gives the energy difference between spin configurations in panel (a) and (b). It is easy to see that $n_{19,18} = n_{19,20} = n_{19,24} = 1$, and $n_{19,14} = -1$. Following the same procedure we end up with,

$$\begin{aligned}\Delta E_{a,b} &= -2(-\mathcal{J}^m_{19,14} + \mathcal{J}^m_{19,18} + \mathcal{J}^m_{19,20} + \mathcal{J}^m_{19,24}),\\ \Delta E_{a,c} &= -2(\mathcal{J}^m_{8,3} + \mathcal{J}^m_{8,7} + \mathcal{J}^m_{8,9} + \mathcal{J}^m_{8,13}),\\ \Delta E_{a,d} &= -2(-\mathcal{J}^m_{7,2} + \mathcal{J}^m_{7,6} + \mathcal{J}^m_{8,3} + \mathcal{J}^m_{8,9} + \mathcal{J}^m_{8,13}\\ &\quad -\mathcal{J}^m_{12,11} - \mathcal{J}^m_{12,13} - \mathcal{J}^m_{12,17}),\\ \Delta E_{a,e} &= \Delta E_{a,b} + \Delta E_{a,d}\\ &= -2(-\mathcal{J}^m_{19,14} + \mathcal{J}^m_{19,18} + \mathcal{J}^m_{19,20} + \mathcal{J}^m_{19,24}\\ &\quad -\mathcal{J}^m_{7,2} + \mathcal{J}^m_{7,6} + \mathcal{J}^m_{8,3} + \mathcal{J}^m_{8,9} + \mathcal{J}^m_{8,13}\\ &\quad -\mathcal{J}^m_{12,11} - \mathcal{J}^m_{12,13} - \mathcal{J}^m_{12,17}).\end{aligned} \tag{S7}$$

Now, assume $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3$, and C_4 are constraints associated with $\Delta E_{a,b}, \Delta E_{a,c}, \Delta E_{a,d}, \Delta E_{a,e}$, respectively. If these constraints are satisfied, i.e., $\Delta E_{a,,b} = \Delta E_{a,c} = \Delta E_{a,d} = \Delta E_{a,e} = 0$, for certain coupling realizations, then they belong to the set $\mathsf{Sat}_{|\mathsf{s}\rangle}$. That is, $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3, \mathsf{C}_4 \in \mathsf{Sat}_{|\mathsf{s}\rangle}$.

To understand this better, consider the case $m = 4$. From Eq. (2), couplings $\mathcal{J}^4_\alpha$ may acquire the values $-2, -1, 0, 1, 2$. In Fig. S2, we provide three examples of random coupling realizations. The spin configuration is the same as in panel (a) of Fig. S1. From Eq. (S7) and Fig. S2, we can see that, only C_3 in panel (a), none in panel (b), and only C_1 in panel (c) are satisfied.

In order to create the subset $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$, we should note that it is not necessarily unique, since we may have different linearly independent constraints that span the same set of conditions in $\mathsf{Sat}_{|\mathsf{s}\rangle}$. In addition to that, the satisfaction of constraints depends on the coupling realizations as well. For instance, if for a given realization, $\mathsf{C}_1, \mathsf{C}_2$ and C_3 are satisfied, trivially from Eq. (S7) (i.e., $\Delta E_{a,e} = \Delta E_{a,b} + \Delta E_{a,d}$), C_4 is automatically satisfied. Therefore, for such cases, C_4 is a linear combination of C_1 and C_3, and one may define the subset $\overline{\mathsf{Sat}}^{(\mathsf{I})}_{|\mathsf{s}\rangle}$ for which $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3 \in \overline{\mathsf{Sat}}^{(\mathsf{I})}_{|\mathsf{s}\rangle}$, but $\mathsf{C}_4 \notin \overline{\mathsf{Sat}}^{(\mathsf{I})}_{|\mathsf{s}\rangle}$. On the other hand, there exist some realizations for which $\Delta E_{a,b} = -\Delta E_{a,d} \neq 0$, but $\Delta E_{a,e} = 0$. Meaning, C_4 is satisfied, however, C_1 and C_3 are not.

The geometrically disjoint constraints may also give rise to different subsets. For instance, from Fig. S1, one can trivially show that the pairs $\mathsf{C}_1, \mathsf{C}_2$ and $\mathsf{C}_1, \mathsf{C}_3$ are each geometrically disjoint, however, C_2 and C_3 are not. Therefore, we could define two different subsets $\overline{\mathsf{Sat}^{\mathsf{g}}}^{(\mathsf{I})}_{|\mathsf{s}\rangle}$ and $\overline{\mathsf{Sat}^{\mathsf{g}}}^{(\mathsf{II})}_{|\mathsf{s}\rangle}$ so that $\mathsf{C}_1, \mathsf{C}_2 \in \overline{\mathsf{Sat}^{\mathsf{g}}}^{(\mathsf{I})}_{|\mathsf{s}\rangle}$ and $\mathsf{C}_1, \mathsf{C}_3 \in \overline{\mathsf{Sat}^{\mathsf{g}}}^{(\mathsf{II})}_{|\mathsf{s}\rangle}$.

These examples further illustrate the difference between $\mathcal{M}$ and $M(\{\mathcal{J}^m_\alpha\})$ in the main text, where $\mathcal{M}$ is associated with the maximum number of linearly independent satisfied constraints, i.e., the cardinality of $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$, while $M(\{\mathcal{J}^m_\alpha\})$ denotes the number of constraints satisfied for a particular realization of coupling constants. Trivially, $M(\{\mathcal{J}^m_\alpha\}) \leq \mathcal{M}$.

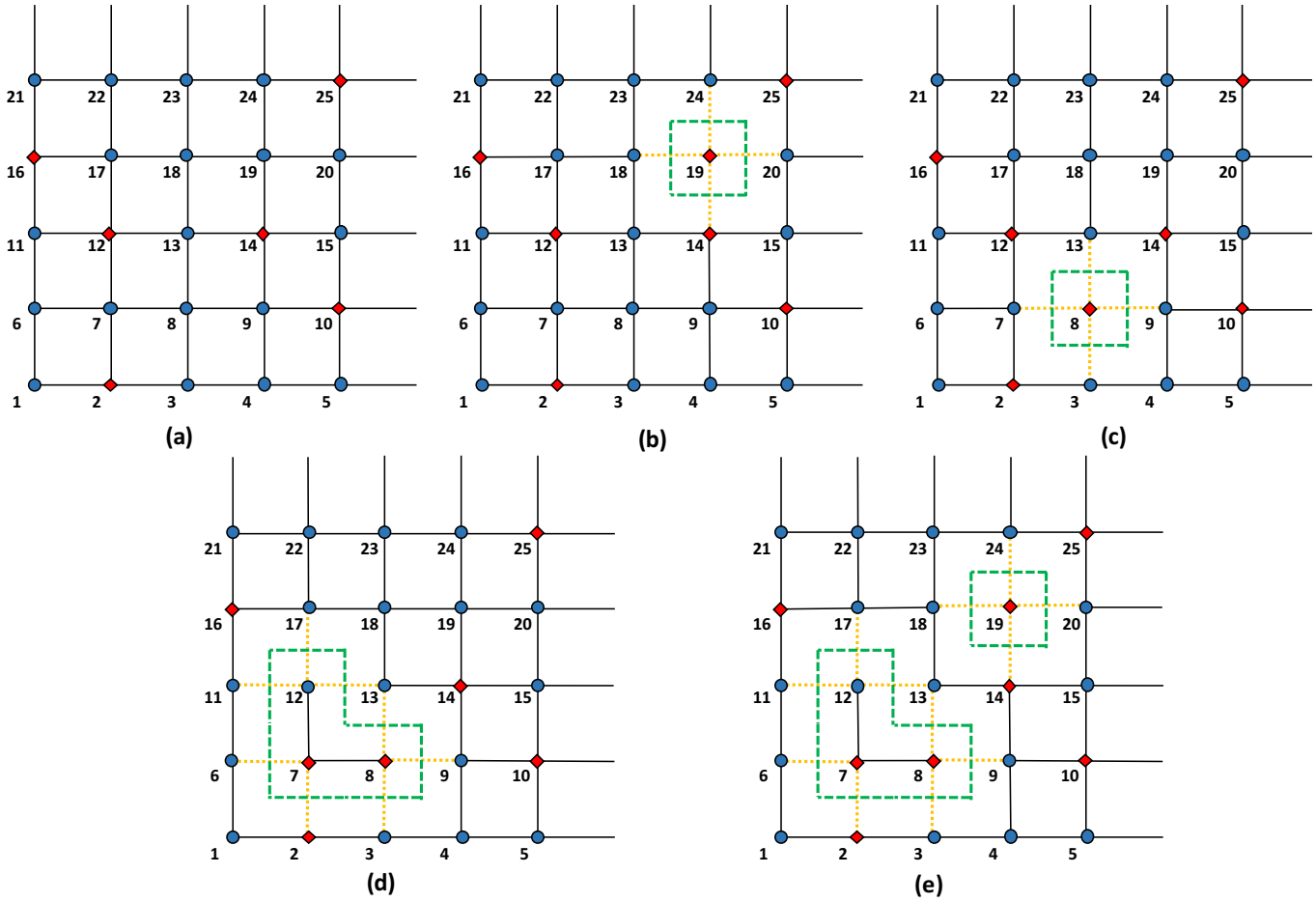

FIG. S1. Graphical representations of the constraints. Panel (a) represents a random spin configuration. Blue solid circles and red diamonds denote spin up and down, respectively. Flipping one or more spins at different sites of panel (a) would result in new spin configurations such as in panels (b) through (e) (e.g., the spin configuration of panel (b) is obtained from flipping the spin at site 19 of panel (a)). The dashed yellow dotted lines represent the links that contribute to the energy difference. The green dashed lines crossing such links correspond to a domain wall.

D. Meaning of Equation (6)

In Eq. (6) of the main text we mentioned that the set $\{|\bar{n}_1\bar{n}_2\cdots\bar{n}_M\rangle\}$ includes all of the spin configurations degenerate with $|\mathsf{s}\rangle$. We also pointed out that it may contain additional states not degenerate with $|\mathsf{s}\rangle$. The latter point is usually associated with the domain walls that are not geometrically disjoint (see section C). To accentuate this consider, e.g., a 5×5 lattice with a given random spin configuration and coupling constants (see panel (a) of Fig. S3), in which $U_{\mathsf{ba}}, U_{\mathsf{ca}}$, and U_{da} are spin flip operators leading, respectively, to zero energy domain walls around the sites 7, 18 and 19 (corresponding to panels (b),(c) and (d)).

From Fig. S3, the domain walls in panel (c) and (d) are not geometrically disjoint, where U_{ca} and U_{da} act on the *nearest neighbor* sites 18 and 19 such that the sign of the link connecting them, is altered by both operators. In such a case, even though the two states $U_{\mathsf{ca}}|\mathsf{a}\rangle \equiv |\mathsf{c}\rangle$ and $U_{\mathsf{da}}|\mathsf{a}\rangle \equiv |\mathsf{d}\rangle$ are degenerate with $|\mathsf{a}\rangle$, the state $U_{\mathsf{da}}U_{\mathsf{ca}}|\mathsf{a}\rangle \equiv |\mathsf{e}\rangle$ (i.e., from panel (e), $U_{\mathsf{ea}} = U_{\mathsf{da}}U_{\mathsf{ca}}$) is not degenerate with $|\mathsf{a}\rangle$. One should note that in general this might not be true. That is, for some coupling realizations the state $|\mathsf{e}\rangle$ can be degenerate with $|\mathsf{a}\rangle$.

By contrast, the two spin flip operators U_{ba} and U_{da} associated with the geometrically disjoint domain walls in panel (b) and (d), respectively, do not alter the signs of any common links. Therefore, the state $U_{\mathsf{da}}U_{\mathsf{ba}}|\mathsf{a}\rangle \equiv |\mathsf{f}\rangle$ (i.e., from panel (f), $U_{\mathsf{fa}} = U_{\mathsf{da}}U_{\mathsf{ba}}$) is degenerate with $|\mathsf{a}\rangle$.

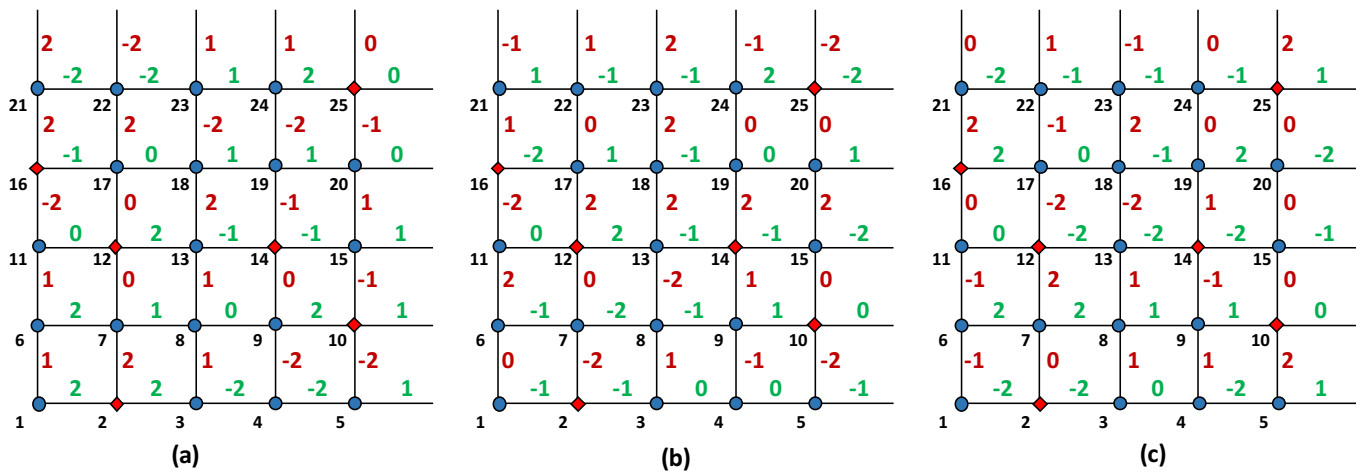

FIG. S2. Three examples of coupling realizations for the binomial model with $m = 4$ (i.e., $\mathcal{J}_\alpha^4 = -2, -1, 0, 1, 2$). The numbers in green (brown) color provide the values of horizontal (vertical) coupling constants.

E. The ground state entropy is bounded by the entropy of a random energy level

In deriving the bound of Eq. (12), we assumed that no information other than the probability distribution $P(\{\mathcal{J}_\alpha^m\})$ is provided. The configuration $|\mathsf{s}\rangle$ that we considered in the main text was an arbitrary random state. We next consider a more sophisticated problem. Suppose that the coupling constants are drawn from a binomial distribution and that once chosen a *ground state configuration* $|\mathsf{s}\rangle$ is given (i.e., the values of the spins s_x at all sites x in this ground state are provided). We then calculate the average of Eq. (8) with the condition that the (otherwise random binomial) coupling constants admit the particular configuration $|\mathsf{s}\rangle$ as *a ground state*. When applicable, the fact that $|\mathsf{s}\rangle$ is a ground state may generally yield nontrivial constraints on the coupling constants $\{J_\alpha^{(k)}\}$ (recall that $\mathcal{J}_\alpha^m \equiv \frac{1}{\sqrt{m}}\sum_{k=1}^m J_\alpha^{(k)}$). In such a situation, given the configuration $|\mathsf{s}\rangle$, we may not simply use the initial binomial distribution for the coupling constants.

We now trivially demonstrate that if the energy density associated with the high temperature limit is unique then Eq. (12) constitutes an upper bound on the average ground state entropy density even if such information was provided for each realization of $\{J_\alpha^{(k)}\}$. This assertion follows as the entropy $S_\ell(\{J_\alpha^{(k)}\})$ associated with any energy $E = E_\ell$ is typically larger than the ground state entropy,

$$S_0 \leq S_\ell. \tag{S8}$$

The proof of Eq. (S8) is rather elementary and relies on a trivial symmetry of the spectrum. Let us denote the two sublattices forming the large bipartite lattice by A and B. If we flip all spins in sublattice A (i.e., $s_{\mathsf{x}\in\mathsf{A}} \to -s_{\mathsf{x}\in\mathsf{A}}$) and do not alter those in sublattice B ($s_{\mathsf{y}\in\mathsf{B}} \to s_{\mathsf{y}\in\mathsf{B}}$), then all nearest-neighbor links (i.e., the products $s_\mathsf{x} s_\mathsf{y}$ for nearest neighbor sites x and y) on the original lattice change their sign, $z_\alpha \to -z_\alpha$. This single sublattice spin inversion constitutes a one-to-one mapping of the Ising spin states, that changes the sign of the total energy ($E \to -E$). We may thus conclude that as a function of the energy E, the entropy density $\mathcal{S} = S(\{J_\alpha^{(k)}\})/N$ for a system with fixed couplings $\{J_\alpha^{(k)}\}$ satisfies the simple relation $\mathcal{S}(E_\ell) = \mathcal{S}(-E_\ell)$ where E_ℓ is the energy of the ℓ-th level. It follows that the energy $E = 0$ is an extremum of the entropy density $\mathcal{S}(E) \equiv \mathcal{S}(E_\ell)$. Consequently, for any fixed couplings $\{J_\alpha^{(k)}\}$,

$$\frac{1}{T} = N\frac{\partial \mathcal{S}}{\partial E} \geq 0. \tag{S9}$$

(The factor of N appears in the above equation since $\mathcal{S}$ is the entropy density). Thus, $E \leq 0$ for any positive temperature T. In what follows we discuss what occurs if there is a *unique* high temperature limit for each set of coupling constants. In such a case, the entropy density $\mathcal{S}(E)$ (averaged over all realization of the coupling constants) is maximal at $E = 0$. The semi-positive definite nature of the derivative in Eq. (S9) implies (as in all common systems satisfying the third law of thermodynamics) that the entropy is lowest at $T = 0$. Since the state $|\mathsf{s}\rangle$ for which we performed the analysis was arbitrary (and corresponds to an energy $E_{|\mathsf{s}\rangle}$ for which the entropy density is greater than or equal to that of the ground state), we see that Eq. (S9) must hold even if information is provided as to the explicit ground state configuration $|\mathsf{s}\rangle$ for each particular realization of the couplings $\{J_\alpha^{(k)}\}$. We thus observe that even if given such additional information, the ground state entropy density must satisfy the bound of Eq. (12).

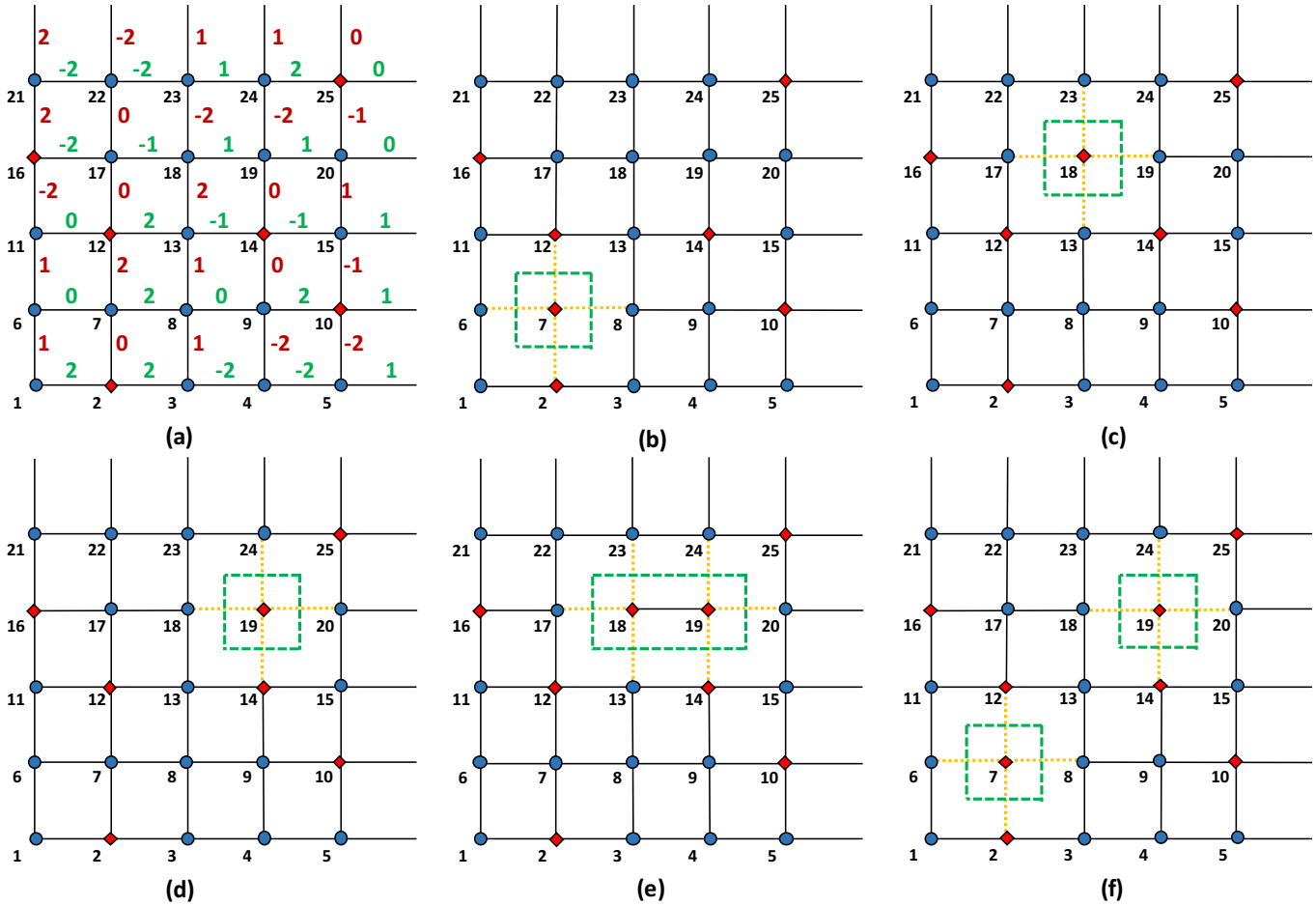

FIG. S3. Panel (a) represents a random spin configuration with some given coupling constants. Blue solid circles and red diamonds denote spin up and down, respectively. The numbers in green (brown) color provide the values of horizontal (vertical) coupling constants. Flipping one or more spins at different sites of panel (a) would result in new spin configurations such as in panels (b) through (f). The dashed yellow dotted lines represent the links that contribute to the energy difference. The green dashed lines crossing such links correspond to a domain wall. Please note that the values associate with different links in each panel is the same as in panel (a).

F. Asymptotic Scaling of the Entropy Density

We now motivate a scaling that the rigorous bound of Eq. (12) suggests Eq. (13) as an approximate asymptotic relation for large N and m. In Section C of this supplemental material, we defined the subset $\overline{\mathsf{Sat}^{\mathsf{g}}}_{|\mathsf{s}\rangle} \subset \overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$ composed of geometrically disjoint constraints. If there are n_g such constraints (or associated zero energy domain walls when these constraints are satisfied) then the degeneracy will be trivially bounded from below by 2^{n_g}. This bound is established by noting that, since no spin is common to two domain walls, all of the spins in each of these n_g domain walls may be flipped independently of all others. When applied to domain walls in $\overline{\mathsf{Sat}^{\mathsf{g}}}_{|\mathsf{s}\rangle}$ then, in the notation of Eq. (6), each binary string of length n_g will correspond to a different configuration that is degenerate with the reference state $|\mathsf{s}\rangle$. This is to be contrasted with the set of zero energy domain walls $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$ for which various binary strings of the form of Eq. (6) may correspond to states that are *not* degenerate with $|\mathsf{s}\rangle$ [22]. As m grows, by Eq. (9), both the number of satisfied constraints and the number of independent zero energy domain walls may diminish as $1/\sqrt{m}$. When fewer walls appear in $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$, it may become increasingly rare for different walls in this subset to share the same lattice sites. If this occurs then, for large m, we will have the asymptotic relation $\overline{\mathsf{Sat}^{\mathsf{g}}}_{|\mathsf{s}\rangle} \sim \overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$. In such a case, in the large N limit, $\mathcal{S} \sim n_g/N \ln 2$. The number n_g and the probability of these zero energy domain walls decay, for $m \gg 1$, as $1/\sqrt{m}$ (or $1/\sqrt{m'}$ for $m' \gg 1$). Similarly, if a finite fraction of the M domain walls in $\overline{\mathsf{Sat}}_{|\mathsf{s}\rangle}$ does not remain geometrically disjoint such that, asymptotically, one may only generate q^M (with $q < 2$) degenerate states (Eq. (6)) given M independent domain walls, then $\mathcal{S} \sim \frac{M}{N} \ln q$. Either way, we anticipate that in the thermodynamic limit that Eq. (13) will hold.

G. One-dimensional Binomial Spin Glass

Let us start with the simplest one-dimensional binomial spin glass system (which by a simple change of variables ($s_{\mathsf{x}} \to s'_{\mathsf{x}} \equiv s_{\mathsf{x}} \prod_{\mathsf{u}<\mathsf{x}} \mathsf{sign}(\mathcal{J}^m_{\mathsf{u},\mathsf{u}+1})$) may be transformed onto a random Ising ferromagnet with couplings $|\mathcal{J}^m_{\mathsf{x}',\mathsf{x}'+1}|$). Here, the ground state energy $E_0 = -\sum_x |\mathcal{J}^m_{\mathsf{x},\mathsf{x}+1}|$. In an open chain of N sites, the lowest excitation consists of identifying the weakest link, $|\mathcal{J}^m_{\mathsf{x}',\mathsf{x}'+1}| \equiv \min_{\mathsf{x}}\{|\mathcal{J}^m_{\mathsf{x},\mathsf{x}+1}|\}$ and flipping all spins $s_{\mathsf{x}} \to -s_{\mathsf{x}}$ for which $\mathsf{x} > \mathsf{x}'$ (or consistently doing the same thing and only flipping all spins to the left of x'); this generates a state that has an energy $E_0 + \Delta E_{\min}$ with $\Delta E_{\min} = 2|\mathcal{J}^m_{\mathsf{x}',\mathsf{x}'+1}|$. (On a periodic chain, we may similarly identify the two weakest links and flip all spins lying between those two links leading to an energy cost $\Delta E_{\min}$ that is twice the sum of the moduli of these two weakest links.) Calculations of the density of states and all ensuing thermodynamic properties are trivial [23]. For instance, the disorder averaged entropy in the low temperature, $T \ll 1$, limit of the binary model is $[S_{m=1}(T)] \sim k_B(\ln 2 + (N-1)(1+2\beta)e^{-2\beta})$, with $\beta = 1/(k_B T)$. The exponential suppression becomes $e^{-2\beta/\sqrt{m}}$ and $e^{-4\beta/\sqrt{m}}$ for odd and even m, respectively. Thus the excitation gap scales as $m^{-1/2}$ (*yet with differently for odds and even* m). By contrast, the low-T entropy of the continuum model is $[S_{m\to\infty}(T)] \sim k_B(\ln 2 + \frac{N-1}{\sqrt{2\pi}}(k_B T - \frac{(k_B T)^3}{8}))$, indicating the vanishing of the spectral gap in the thermodynamic limit. In that limit, these lowest excitations differ, relative to the ground state, by an extensive number of flipped spins.

H. Distribution of excitations

Given any *ground state* configuration on a hypercubic lattice in d dimensions, one may compute the probability distribution for excitations of energy $\Delta E_{\mathsf{x}} = |\Delta E_{\mathsf{x}}| = 2\sum_{\mathsf{y}} n_{\mathsf{xy}} \mathcal{J}^m_\alpha$ generated by flipping a single spin x. Here, the sum is over all sites y that are nearest neighbor of site x and $n_{\mathsf{xy}} = -\mathsf{sign}(s_{\mathsf{x}} s_{\mathsf{y}}) = \pm 1$. Given the probability distribution for the links $\{\mathcal{J}^m_\alpha\}$, one may compute the probability distribution associated with a finite sum of these links $2\sum_{\mathsf{y}} n_{\mathsf{xy}} \mathcal{J}^m_\alpha$ in the ground state. The latter sum is that over a finite number of links (with bounded mean and variance) and thus for any $\epsilon > 0$ (no matter how small), the probability that $|\Delta E_{\mathsf{x}}| < \epsilon$ is strictly smaller than unity. In order for the system to have a spectral gap that is greater than ϵ, it must be that *for each of the N lattice sites* x, the energy penalty $|\Delta E_{\mathsf{x}}| > \epsilon$. Given that the condition $|\Delta E_{\mathsf{x}}| > \epsilon$ must, in the thermodynamic limit, be satisfied an infinite number of times, while for any single x the probability that this condition is satisfied is strictly smaller than one, it is essentially impossible to have a gap larger than any arbitrary positive number ϵ. From this, it follows that the gapless local excitations must be appear. If the local energy penalties in the ground state were independent of one another then the probability that all local flips result in an energy penalty larger than ϵ would the product of the probabilities of having $|\Delta E_{\mathsf{x}}| > \epsilon$ for all sites x. Although the local flip are not independent of one another (since they all relate to flips relative to the same special state- the ground state), it seems highly unlikely $|\Delta E_{\mathsf{x}}| > \epsilon$ for all x when the probability of having a local energy penalty larger than ϵ for any single x is strictly smaller than one.

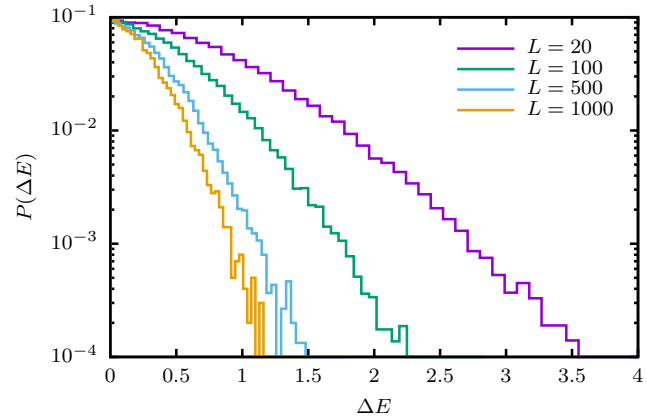

FIG. S4. Distribution of defect energies $|\Delta E|$ for the 2D Gaussian system and a number of different system sizes. The curves collapse if rescaled by $L^{-\theta}$ with the value $\theta \approx -0.28$ of the stiffness exponent.

We now explicitly discuss a measure that, in general dimensions, may provide physical insight – the *distribution* of such individual defect energies (i.e., the distribution of domain wall energies in our binomial Ising spin system). In Fig. S4, we plot this distribution in the continuous $m = \infty$ Gaussian limit. If $f(\epsilon, \tilde{l})$ denotes the cumulative probability that the energy penalty of a domain wall (of size $\tilde{l}$) is smaller than ϵ, then the probability that amongst $\mathcal{N}_{\tilde{l}}$ *independent* domain walls, no singe domain wall entails an energy cost lower than ϵ will be bounded from above by $e^{-f(\epsilon,\tilde{l})\mathcal{N}_{\tilde{l}}}$ as we briefly elaborate on now. Since, by definition, $f(\epsilon, \tilde{l})$ is the cumulative probability that the energy cost of a random wall of size $\tilde{l}$ is smaller than ϵ (i.e., $Prob.(|\Delta E| \le \epsilon) = f(\epsilon, \bar{l})$), the probability that amongst $\mathcal{N}_{\tilde{l}}$ *independent* domain walls, we explicitly have that the probability that no single domain wall has an energy cost larger than ϵ is, trivially, $[Prob.(|\Delta E| > \epsilon)]^{\mathcal{N}_{\tilde{l}}} = (1 - f(\epsilon, \tilde{l}))^{\mathcal{N}_{\tilde{l}}} \le e^{-\mathcal{N}_{\tilde{l}} f(\epsilon,\tilde{l})}$ (where we invoked $e^{-f} \ge (1-f)$ for all $f \ge 0$). For small $f \to 0^+$ (associated with $\epsilon \to 0^+$ in $d \ge 3$), this general inequality is replaced by an equality (i.e., $[Prob.(|\Delta E| > \epsilon)]^{\mathcal{N}_{\tilde{l}}} = e^{-\mathcal{N}_{\tilde{l}} f(\epsilon,\tilde{l})}$).

Thus, if the area ($d = 2$) or volume ($d = 3$) of the entire lattice is $||\Lambda||$, then whenever the sum

$$\lim_{\epsilon\to 0^+} \lim_{\tilde{l}_0\to\infty} \lim_{N\to\infty} \sum_{||\Lambda||^{1/d} - \tilde{l}_0 \ge \tilde{l} \ge \tilde{l}_0} f(\epsilon, \tilde{l}) \mathcal{N}_{\tilde{l}} = \infty \quad \text{(S10)}$$

then gapless (or degenerate) states of diverging $\tilde{l}$ may

appear. This is so because flipping all of the spins links one ground state to its conjugate. The inequality $||\Lambda||^{1/d} - \tilde{l}_0 \geq \tilde{l} \geq \tilde{l}_0$ in Eq. (S10) means that the an extensive number of spin flips is needed to connect a given spin configuration to either of the two members of the degenerate ground state pair.

Since $\theta_{d=2} < 0$ then (as is further underscored in the full distribution of Fig. S4), in two dimensions nearly all large domain walls entail a *vanishing* energy penalty. In $d = 2$, $\lim_{\epsilon \to 0^+} \lim_{\tilde{l} \to \infty} f(\epsilon, \tilde{l}) = 1$ and the probability of obtaining, in the thermodynamic limit, degenerate states that differ by an extensive number of flipped spins is unity. The existence of gapless states in $d = 2$ is hardly surprising; such gapless states may be trivially constructed by the insertion of random domain walls of divergent size into a ground state. Indeed, in $d = 2$ (where the typical energy cost $\mathcal{O}(\tilde{l}^{\theta})$ vanishes as $\tilde{l} \to \infty$), knowledge of the detailed distribution of the energy cost as a function of the domain wall size $\tilde{l}$ is unnecessary for establishing gapless states. However, in $d \geq 3$ (where $\theta_d > 0$), the lowest energy states are related to *the asymptotic low energy limit* of the domain wall energy distribution (a distribution that, in these higher dimensions, is associated with a divergent average energy $\mathcal{O}(\tilde{l}^{\theta_d})$ when $\tilde{l} \to \infty$). A gap (for states that differ from one another by an extensive number of flipped spins) is potentially possible if the sum of Eq. (S10) vanishes. Thus, we stress that in $d \geq 3$, knowledge of the cumulative probability distribution $f(\epsilon, \tilde{l})$ can be of paramount importance. We reserve the analysis of the $d = 3$ domain wall energy distribution for future work.

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[23] In order to attain any state, a number of links $\{\mathcal{J}_\alpha^m\}$ must be flipped relative to a ground state. The probability for obtaining a particular energy amounts to a convolution on the probability distribution of the links. The latter convolution readily becomes a trivial product after Fourier transformation.